MATH 336: Homework #11 (Extra Credit)

Due: Monday, May 2, 2016

Solve the below problems concerning partial differential equations. This assignment is optional, but can be used as a substitution for a previous homework grade. Please bring it to my office by **5 pm on Monday, May 2.** Note all of this material is covered on Exam 3, so I highly suggest attempting it, especially before May 6th. Feel free to see me with any questions about any of the material. ES denotes the online lecture notes.

1. (20 points) Consider the one-dimensional constant coefficient diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}. \tag{1}$$

Show that equation (1) satisfies **superposition**. Namely, show that if $c_1(x,t)$ and $c_2(x,t)$ both satisfy (1) (i.e. they are solutions), then any linear combination $\alpha_1 c_1(x,t) + \alpha_2 c_2(x,t)$ does as well, for any $\alpha_1, \alpha_2 \in \mathbb{R}$. This shows that the solution set of the homogeneous diffusion equation (in fact it is true of any linear homogeneous PDE) is a vector space. This is the most important property of linear equations, and allows a general theory to be developed, and is in direct contrast with nonlinear equations, which in general do not satisfy superposition.

2. (10 points) (ES, p.200, #1) Problem 1 in the PDE2 section of the notes (end of chapter 3).

3. (15 points) (ES, p.200, #2) Problem 2 in the PDE2 section of the notes (end of chapter 3).

4. (15 points) (ES, p.200, #4) Problem 4 in the PDE2 section of the notes (end of chapter 3).

5. (20 points) Show that the following is a particular solution of the one-dimensional heat equation (1) on $(-\infty, \infty)$:

$$c_0(x,t) = \frac{C}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}, \tag{2}$$

where C is any real constant. This is a fundamental solution of the heat equation, and takes the form of a **Gaussian** function. Note that if you have seen some probability theory, you may notice a connection between (2) and the normal distribution. This is no coincidence.

6. (20 points) (ES, p.203, #6) Problem 6 in the PDE3 section of the notes (end of chapter 3).

   Hint: For part (b), you will at some point need to use the Gaussian solution (2), after you reduce the *advection-diffusion* equation to a standard 1D *pure* diffusion equation.