MATH 350: Homework #9

Due: Thursday, November 29, 2018

Solve the below problems concerning diagonalizability, invariant subspaces, and the Cayley-Hamilton theorem. A (possibly improper) subset of them will be graded. All calculations should be done analytically.

1. Define the linear operator $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

   Is T diagonalizable? If so, find an ordered basis β such that $[T]_\beta$ is diagonal, and find $[T]_\beta$.

2. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

   Find an expression for A^n, for an arbitrary positive integer n.
   Hint: Attempt to diagonalize, and note that if

$$A = Q^{-1}BQ,$$

   then

$$\begin{aligned} A^2 &= (Q^{-1}BQ)(Q^{-1}BQ) \\ &= Q^{-1}B^2Q. \end{aligned}$$

   Generalize this for any positive integer n.

3. Fix $f(t)$ as a polynomial with coefficients from F. Let $T : V \to V$, where V is a vector space over F. Recall that $f(T)$ is a linear operator from V to V. Show that

$$f(T)T = Tf(T),$$

   i.e. that T and $f(T)$ commute with one another. Note that $f(T)T$ is a linear operator on V defined by

$$(f(T)T)(v) := f(T)(T(v)),$$

   and similarly for $Tf(T)$.

4. Consider the following $k \times k$ matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$f_A(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k).$$

Hint: Use induction on the size of the matrix k, and expand the determinant along the first row.

5. Consider the linear map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A.$$

Find the T-cyclic subspace generated by

$$z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note: using my notation from class, I am asking you to find $W = Z(z, T)$.

6. Let A be a square matrix of size n. Prove that

$$\dim(\operatorname{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$

Hint: Use the Cayley-Hamilton theorem.

7. Let V be a **two-dimensional** vector space, and $T \in \mathcal{L}(V)$. Show that if V is not itself a cyclic subspace (that is, there exists no $v \in V$ such that $V = Z(v, T)$), then

$$T = cI_V,$$

for some $c \in F$. Here I_V is the identity operator on V.
Hint: This requires some finesse. Note that we can form a basis $\beta = \{v_1, v_2\}$ for V. What does the assumption say about $T(v_1)$ and $T(v_2)$? Relate this to eigenvalues and eigenvectors.