MATH 350: Homework #8

Due: Thursday, November 8, 2018

Solve the below problems concerning determinants and eigenvalues. A (possibly improper) subset of them will be graded. All calculations should be done analytically.

1. Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

2. A matrix $M \in M_{n\times n}(C)$ is called **skew-symmetric** if $M^T = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. Is the same true if n is even? Prove or give a counterexample.

3. The goal of this exercise is to compute determinants of block diagonal matrices. That is, square matrices of the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are square matrices, and 0 is a matrix consisting entirely of zeros. We proceed in steps.

(a) Suppose that $A \in M_{n\times n}(F)$. Using mathematical induction and the definition of the determinant, prove that

$$\det \begin{pmatrix} A & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} = \det(A)$$

Note that the below analogous formula holds as well for $B \in M_{m\times m}(F)$ (you do not have to prove this):

$$\det \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & B \end{pmatrix} = \det(B)$$

(b) With A and B as above, use (a) to show that

$$\det\begin{pmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{pmatrix} = \det(A)\det(B).$$

Hint: Note that by matrix multiplication we may factor

$$\begin{pmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{pmatrix} = \begin{pmatrix} A & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix}\begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & B \end{pmatrix}.$$

(c) Prove that for A and B square

$$\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B).$$

Here 0 is a matrix of zeros of the right size. *Hint:* Work in two cases: A is not invertible, and A is invertible. The first case is simpler. For the case where A^{-1} exists, use a factorization similar the one suggested in part (b).

4. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of $\mathbb{R}^n$ containing n distinct vectors. Let $B \in M_{n\times n}(\mathbb{R})$ be the matrix having u_j as its jth column:

$$B = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}.$$

Prove that β is a basis for $\mathbb{R}^n$ if and only of $\det(B) \neq 0$.

5. **True or False**: Every linear operator on an n-dimensional vector space has n distinct eigenvalues. Prove or provide a counterexample.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + 3y \\ -10x + 9y \end{pmatrix}$$

Find the eigenvalues of T and an ordered basis β for $\mathbb{R}^2$ such that $[T]_\beta$ is diagonal. Provide the change of coordinates matrix Q that takes T from the standard ordered basis of $\mathbb{R}^2$ to β.

7. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Find the eigenvalues of T and an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.