MATH 350: Homework #6

Due: Thursday, October 25, 2018

Solve the below problems concerning matrices, isomorphisms, and coordinates. A (possibly improper) subset of them will be graded. All calculations should be done analytically.

1. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$. Recall that T_0 is the zero transformation of V:

$$T_0(x) = 0,$$

for all $x \in V$.

2. Let V be a finite dimensional vector space, and let $T : V \to V$ be linear.
   (a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \bigoplus N(T)$.
   (b) Prove that $V = R(T^k) \bigoplus N(T^k)$ for some positive integer k.

3. **True or False**: $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is a vector space isomorphism, where

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2.$$

Provide justification.

4. Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \middle| a, b, c \in F \right\}.$$

Construct an isomorphism between V and F^3.

5. Suppose that V is finite dimensional, and $S, T \in \mathcal{L}(V)$. Prove that $ST = I_V$ if and only if $TS = I_V$.

6. (**Hard**) Prove that if there exists linear map on V (i.e. $T : V \to V$, T linear), such that both the null space and range of T are finite-dimensional,

then V is finite dimensional.
Hint: Write down bases for $N(T)$ and $R(T)$. Note that every element in the basis for $R(T)$ is of the form $T(x_i)$, some $x_i \in V$ (this is just the definition of the range). Now, for any $x \in V$, $T(x) \in R(T)$. Proceed from there to show that x can be written as a finite linear combination of vectors.