MATH 350: Homework #5

Due: Thursday, October 18, 2018

Solve the below problems concerning coordinates and matrix representations of linear transformations (mainly Section 2.2). A (possibly improper) subset of them will be graded. All calculations should be done analytically.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ y \\ 2x + y \end{pmatrix}.$$

   Let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

   Compute $[T]_\beta^\gamma$.

2. Recall that the set of 2×2 symmetric matrices is a vector space with basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

   Find the coordinates $[A]_\beta$ of the symmetric matrix A relative to β, where

$$A = \begin{pmatrix} -2 & -1 \\ -1 & \pi \end{pmatrix}.$$

3. **True or False**: $\mathcal{L}(V, W) = \mathcal{L}(W, V)$ for all vector spaces V and W over the same field F. If true, prove. If false, give a counterexample.

4. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\beta = \{1, x, x^2\}$$

$$\gamma = \{1\}$$

(a) Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^T$. Compute $[T]_\alpha$.

(b) Define $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

Compute $[T]_\beta^\alpha$.

(c) Define $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = \mathrm{tr}(A)$. Compute $[T]_\alpha^\gamma$.

(d) Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$, compute $[A]_\alpha$.

(f) If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$.

(g) For $a \in \mathbb{R}$, compute $[a]_\gamma$.

5. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by

$$T(x) = [x]_\beta.$$

Prove that T is linear. Is T one-to-one (injective)?