MATH 350: Homework #4

Due: Monday, October 8, 2018

Solve the below problems concerning linear transformations (mainly Sections 2.1 and 2.2). A (possibly improper) subset of them will be graded. All calculations should be done analytically. Note you do not need to do the problems in **red** to hand in for Monday 10/8. But they are good practice for the exam.

1. Let $T : F^4 \to F^2$ be a linear map. Suppose that

$$N(T) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5x_2,\ x_3 = 7x_4\}.$$

   Show that T is surjective.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exists scalars $a, b, c \in \mathbb{R}$ such that

$$T(x, y, z) = ax + by + cz.$$

   for all $x, y, c \in \mathbb{R}$. Can you generalize this to $T : F^n \to F$? What about linear transformations between F^n and F^m?

   For the following exercises, we need the following definition.

   Definition. Let V be a vector space, and W_1, W_2 subspaces such that $V = W_1 \bigoplus W_2$. A function $T : V \to V$ is called a **projection on $\mathbf{W_1}$ along $\mathbf{W_2}$**, if, for $x = x_1 + x_2$, with $x_1 \in W_1, x_2 \in W_2$, we have $T(x) = x_1$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$.

   (a) Find a formula for $T(x, y)$, where T represents the projection on the y-axis along the x-axis.

   (b) Find a formula for $T(x, y)$, where T represents the projection on the y-axis along the line $L = \{(a, a) \mid a \in \mathbb{R}\}$.

4. Let $T : V \to V$ be a projection on W_1 along W_2. Prove that T is linear.

5. Let T be as in Problem 4. Show that $W_1 = R(T)$ and $W_2 = N(T)$.

6. Recall that the set of 2×2 symmetric matrices is a vector space with basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Find the coordinates $[A]_\beta$ of the symmetric matrix A relative to β, where

$$A = \begin{pmatrix} -2 & -1 \\ -1 & \pi \end{pmatrix}.$$

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x + 2y \\ 10y - x \end{pmatrix}.$$

Let β be the standard basis for $\mathbb{R}^2$, and

$$\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Note that γ is a basis for $\mathbb{R}^3$. Compute the matrix representation $[T]_\beta^\gamma$ of T with respect to bases β and γ.