MATH 350: Homework #3

Due: Thursday, October 4, 2018

Solve the below problems concerning sums and direct sums and linear transformations (mainly Sections 1.6 and 2.1). A (possibly improper) subset of them will be graded. All calculations should be done analytically

1. Let $U_1, U_2, \ldots, U_m$ be subspaces of a vector space V. Show that $W := U_1 + U_2 + \cdots + U_m$ is a subspace of V.

2. Let U_1 and U_2 be finite-dimensional subspaces of a vector space V, and let $V = U_1 + U_2$. Deduce that V is a direct sum of U_1 and U_2 if and only if

$$\dim(V) = \dim(U_1) + \dim(U_2).$$

3. Let $V = M_{2\times 2}(F)$, and

$$U_1 = \left\{ \left(\begin{array}{cc} a & b \\ c & a \end{array} \right) \middle| \; a, b, c \in F \right\},$$
$$U_2 = \left\{ \left(\begin{array}{cc} 0 & a \\ -a & b \end{array} \right) \middle| \; a, b \in F \right\}.$$

Show that U_1 and U_2 are subspaces of $M_{2\times 2}$, and find the dimensions of $U_1, U_2, U_1 + U_2$, and $U_1 \cap U_2$.
Note: we did not prove it, but it is not hard to show that the intersection of subspaces is a subspace, so the final dimension is well-defined.

4. In $\mathbb{R}^4$, let

$$W_1 = \{(a, b, 0, 0) \,|\, a, b \in \mathbb{R}\},$$
$$W_2 = \{(0, 0, c, 0) \,|\, c \in \mathbb{R}\},$$
$$W_3 = \{(0, 0, 0, d) \,|\, d \in \mathbb{R}\}.$$

Show that $\mathbb{R}^4 = W_1 \bigoplus W_2 \bigoplus W_3$.

5. Define $T : P_2(\mathbb{R}) \to P(\mathbb{R})$ by

$$T(f(x)) = xf(x) + f'(x).$$

Show that T is a linear transformation. Is it one-to-one and/or onto? Provide justification.

6. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1,0) = (1,4)$, and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?

7. Is $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (1,y)$ linear? Why or why not?

8. Recall that $P(\mathbb{R})$ is the set of all polynomials with coefficients in $\mathbb{R}$. Define

$$T : P(\mathbb{R}) \to P(\mathbb{R}),$$
$$T(f(x)) = \int_0^x f(t)\,\mathrm{d}t.$$

Prove that T is linear and one-to-one, but not onto.