MATH 350: Homework #10

Due: Wednesday, December 12, 2018

Solve the below problems concerning Jordan canonical form and inner product spaces. A (possibly improper) subset of them will be graded. All calculations should be done analytically.

1. Let $T \in \mathcal{L}(V)$, and let λ be an eigenvalue of T. Show that K_λ is a T-invariant subspace of V. Recall that K_λ is the generalized eigenspace of T corresponding to λ:

$$K_\lambda = \{x \in V \mid (T - \lambda I_V)^p(x) = 0, \text{some } p \in \mathbb{N}\}.$$

2. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by

$$T(A) = 2A + A^T.$$

   Find a Jordan canonical basis of $M_{2\times 2}(\mathbb{R})$, and the corresponding Jordan canonical form of T.

3. Repeat Problem 2 for the matrix A given by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

   That is, find the Jordan form (and Jordan basis) for A. Find a change of coordinates matrix Q that transforms A into the Jordan form of A. *Hint:* Remember, A is the matrix representation with respect to the standard ordered basis.

4. Let V be the (real) subspace of real-valued functions on $\mathbb{R}$ defined by

$$V = \text{span}(\{e^t, te^t, t^2e^t, e^{2t}\}).$$

   Define $T : V \to V$ by

$$T(f) = f'.$$

   Find the Jordan form of T, and a Jordan canonical basis for V.

5. Fix $A \in M_{2\times 2}(\mathbb{C})$ as

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Show that

$$\langle x, y \rangle := xAy^*$$

is an inner-product on $\mathbb{C}^2$. Compute $\langle x, y \rangle$ for $x = (1, i)$ and $y = (1+i, 3-2i)$. Recall that y^* denotes the **conjugate transpose**, or **adjoint**, of the vector $y \in \mathbb{C}^2$.

6. Let V be an inner product space, and $T : V \to V$ is a given linear operator. Suppose that $||T(x)|| = ||x||$ for all $x \in V$. Show that T is injective.

7. Consider the subset $S \subset M_{2\times 2}(\mathbb{R})$ defined by

$$S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}.$$

With respect to the Frobenius norm on $M_{2\times 2}(\mathbb{R})$, find an orthonormal basis β for $W := \text{span}(S)$. Find the coordinates (called the Fourier coefficients) of A with respect to the basis β, where

$$A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}.$$

8. Let V be a finite-dimensional inner product space. Show that for any subspace U of V,

$$V = U \bigoplus U^{\perp}.$$

Recall that $U^{\perp}$ is the orthogonal complement of U. *Hint:* Use Gram-Schmidt orthogonalization.