MATH 336: Homework #2

Due: Tuesday, September 26, 2017

Solve the below problems concerning difference equations. A (possibly improper) subset of them will be graded. All calculations should be done analytically, unless marked with an (M). (M) problems require the use of MATLAB. ES denotes the online lecture notes.

1. (30 points) (ES, p.24, #4) (M) These are measurements of populations obtained in a laboratory experiment using insects:

$$P_0 = 0.97, P_1 = 1.52, P_2 = 2.31, P_3 = 3.36, P_4 = 4.63, P_5 = 5.94$$
$$P_6 = 7.04, P_7 = 7.76, P_8 = 8.13, P_9 = 8.3, P_{10} = 8.36$$

   (a) Plot these points, to convince yourself that they are (roughly) consistent with a logistic model. You may use these commands:

```
times=[0 1 2 3 4 5 6 7 8 9 10];  %times=0:1:10 is shorter
Ps=[0.97 1.52 2.31 3.36 4.63 5.94 7.04 7.76 8.13 8.3 8.36];
plot(times, Ps)
```

   (b) Now estimate the parameters r and K in $\Delta P = rP(1 - P/K)$ by performing the following steps (there are much better methods for estimating sigmoidal functions; we are just being intuitive here):

      (i) first give a guess K by looking at the graph (recall what K represents);

      (ii) then, using that $P_0 \ll K$, approximate $\Delta P_0/P_0$ by r; what value of r do you obtain?

   Plot the data and iteration, starting at the same P_0 with the values that you obtained.
   Finally, using some trial and error, see if you can get a better fit.
   Hint: All of this problem can be completed with the below code (saved in, say, *hw2Prob1.m*), where certain statements are omitted:

```
times=0:1:10;
Ps=; %enter data as a vector

K=;   %obtain from looking at graph of data
```

```
r1=;  %Follow directions in (ii) to estimate r1

% Approximate using r1 from above
N_time=length(times);
P_logistic_first=zeros(1,N_time);
P_logistic_first(1)=Ps(1);  % same initial conditions

for i=1:(N_time-1)  % main update loop
    P_logistic_first(i+1)=;  %Add update step here to solve difference equation
end

% Try a second r (chosen close to r1)
r2=;  %Guess a value close to r1 obtained above
P_logistic_second=zeros(1,N_time);
P_logistic_second(1)=Ps(1);  % same initial conditions

for i=1:(N_time-1)  % main update loop
    P_logistic_second(i+1)=; %Analogous to previous "for" loop
end

% Now plot the data and the two models (difference equations with different r values)
figure(1)
plot(times,Ps,'-ob','LineWidth',2);
hold on;
plot(times,P_logistic_first,'--*r','LineWidth',2);
plot(times,P_logistic_second,'-xk','LineWidth',2);
legend('data','logistic (r=)', 'logistic (r=)');  %add legend entries for r values obtained
xlabel('n');
ylabel('P_{n}');
```

Note that the code will not run without all statements completed. You can either remove the $r2$ part on your first run (to find the value of $r1$), or simply guess any value for $r2$ you'd like, and later adjust.

2. (20 points) Consider the predator-prey system introduced in class:

$$N_{t+1} = 2N_t e^{-P_t}$$
$$P_{t+1} = N_t(1 - e^{-P_t})$$

Note that this is a special case of the model presented on the course site (see the Systems example PDF on the Course Calendar).

(a) Find all steady state(s) (N_*, P_*).
(b) Find the Jacobian matrix $A(N_*, P_*)$ of each steady state.
(c) Write down the characteristic equation of each $A(N_*, P_*)$.
(d) Find the eigenvalues of each $A(N_*, P_*)$.
(e) Using part (d), determine the stability of each steady state (N_*, P_*).

3. (30 points) Consider the difference equation

$$x_{n+1} = 1 - x_n^2$$

Note that we do **NOT** consider the equation in terms of populations, but rather for illustrative purposes. That is, we will consider x_n such that $x_n < 0$.

(a) Find all steady states of the equation, and classify them (if possible) as stable or unstable.

(b) (M) Plot, on the same set of axes, the trajectories satisfying $x_0 = -0.5, 0.5, 1.5$ for $N = 50$ iterations. How do the iterates appear to behave as $n \to \infty$? Note that the solution cannot be found analytically, but can be easily simulated using a *for loop*. For example, for $x_0 = -0.5$, your code should look as below:

```
%first 3 lines are initialization
N=50;  %number of iterates
x1=zeros(1,N);
x1(1)=-0.5; %initial condition

%main for loop to solve the difference equation
for i=1:N-1
    x1(i+1)=1-x1(i)^2;
end

%add analogous statements for the 2 other initial conditions
%also add plotting
```

You should plot x1, x2 , and x3 (x2 and x3 can use the same code, but with different initial conditions) on *the same set of axes*. See the code above, and/or the code in the Review HW.

(c) Find all possible 2-periodic orbits of the equation. Note that a 2-periodic orbit consists of a pair of points $\{x_0, x_1\}$ such that $x_0 \to x_1 \to x_0 \to \ldots$ under the action of the map. *Hint:* Recall that fixed points will show up as solutions here, even though they are **NOT** 2-periodic orbits. But they will help you factor a 4th-order polynomial...

(d) Find the stability of the periodic orbit(s) found in part (c).

4. (10 points) (ES, p.27-28, #1) Problem 1 in the SDE4 section in the notes (end of chapter 1).

5. (10 points) (ES, p.28, #2) Problem 2 in the SDE4 section in the notes (end of chapter 1).