MATH 336: Homework #12

Due: N/A (this is just practice for Exam 3)

For practice, solve the below problems concerning partial differential equations. Any material presented here will be fair game for Exam 3. All calculations should be done analytically. ES denotes the online lecture notes.

1. (20 points) (ES, p.202, #4) Problem 4 in the PDE3 section of the notes (end of chapter 3).

   Hint: You will need to re-derive (essentially) the separation of variables method on the new interval $[0, \frac{\pi}{2}]$, with slightly different boundary conditions. Your answer in part (b) should depend on **two** constants (which should have some conditions on them). The conditions $c(0,0) = 12$ and $\frac{\partial c}{\partial t}(0,0) = -50$ allow you to determine these constants.

2. Consider a bacteria population undergoing diffusion in a thin tube of length π. Outside of the tube, the bacteria are immediately removed, so that here $c(x,t) \equiv 0$. **In addition, the bacteria are also growing exponentially, with rate constant** α. This last assumption makes the problem different from the case discussed in class, but the method of solving is completely analogous.

   (a) Assuming that the tube can be described with one spatial dimension, write an appropriate boundary value problem (PDE and boundary conditions) which describes the above situation.

   (b) Find a non-zero, bounded **separable** solution. Note that you will be able to deduce a specific form of your separation constant (and hence your solution) in order to satisfy all of the given "constraints."

3. Suppose a bacteria population and nutrient both undergo diffusion in $\mathbb{R}^2$. Denote the bacteria concentration by $n(x,t)$, and the nutrient concentration by $c(x,t)$. Suppose that the bacteria consumes the nutrient, and reproduces at a rate of $K(c(x,t))$, and that α units of nutrient need to be consumed to produce **one** bacteria.

   (a) Write down a **system** of PDEs which describe the dynamics of the bacteria and nutrient concentrations. *Note:* The bacteria and nutrient molecules have different sizes, so we shouldn't expect them to have the same diffusion constant.

(b) Suppose now that the rate of nutrient consumption is governed by Michaelis-Menten kinetics. Use this (additional) assumption to modify the above system of PDEs.

4. Suppose a bacterial population undergoes diffusion and chemotaxis in a thin tube along the x-axis, with endpoints $x = 0$ and $x = L$. Suppose the left and right ends of the tube are closed, so that there is **no flux through the boundaries**. For a discussion on this, see Section 3.2.5 in the notes. The chemical potential function describing the chemoattractant is given by

$$V(x) = \frac{1}{1+x^2},$$

and the corresponding rate constant is α. The tube is homogeneous, so that the diffusion term is constant: $D(x) \equiv D$.

(i) Write an expression for the net flux $J(x,t)$ describing the movement of bacteria.

(ii) Find the boundary condition at $x = 0$. Note that this should be an equation relating $c(0,t)$ and $\frac{\partial c}{\partial x}(0,t)$. Being closed (i.e. having no net flux) at the boundary means that $J(0,t) = 0$. Using your expression from (i), relate this to c and (some of) its partial derivatives, **at** $\mathbf{x = 0}$.

(iii) Similarly, find the boundary condition at $x = L$. Again, this should be an equation relating $c(L,t)$ and $\frac{\partial c}{\partial x}(L,t)$.

5. (ES, p.202, #3) Problem 3 in the PDE3 section of the notes (end of chapter 3). *Hint:* Closed at the endpoint $x = 1$ means that there is **no flux** there, i.e. that $J(1,t) = 0$. Since $J = -D\frac{\partial c(x,t)}{\partial x}$ for diffusion, this implies that $\frac{\partial c(1,t)}{\partial x} = 0$. In terms of your separable solution, this is equivalent to (check this!) $X'(1) = 0$. This is one boundary condition; you should find the other. Note that an an outside bacterial density of $c = 5$ tells you something about the open-end boundary. You should rescale to bring the equation to homogeneous boundary conditions (i.e. $c = 0$). See the top of page 178 for a discussion of this (translating to $\tilde{c}(x,t) = c(x,t) - 5$, essentially).

6. Suppose a population of bacteria, with density $c(x,t)$ (in one dimension) evolves according to the following PDE:

$$\frac{\partial c}{\partial t} = -\frac{\partial (cV')}{\partial x},$$

where $V(x) = \frac{x^2}{1+x^2}$.

(a) Sketch a plot of $V(x)$.

(b) Is the motion transport, chemotaxis, or diffusion?

(c) Describe, in one short sentence, what you think the bacteria are doing. For example, "They moves west at 2 mph," or "they move randomly," or "they move towards a food source located at $x = -4$." You are **not** being asked to solve any equations or do any in depth analysis.

7. Suppose that bacteria move on a plate in **two-dimensional** space. Furthermore, suppose that they are transported by an air current moving 5 m/s eastward, and grow exponentially with a doubling time of 1 hour. Also, they move randomly with of a diffusion coefficient of 10^{-3}. Provide a differential equation for $c(x,t)$, the density of the bacterial population, that models this situation. Note that you do **not** need to solve anything; the point is to get you used to "translating" from word descriptions to mathematical equations.