Linearization of autonomous first order equations

Consider the autonomous ODE

$$\frac{dy}{dt} = f(y), \tag{1}$$

and suppose that y_* is an equilibrium point for this equation, i.e., that $f(y_*) = 0$. *Suppose further that $f'(y_*)$ is not zero.* When this is true, the value of $f'(y_*)$ tells us a lot about the nature of the equilibrium point and the behavior of solutions near it.

Qualitative information from $f'(y_*)$. In this section we essentially repeat the discussion of "linearization" on pages 85–86 of Blanchard, Devaney, and Hall; this addresses the question: what can we learn from the sign of $f'(y_*)$, that is, from whether $f'(y_*) > 0$ or $f'(y_*) < 0$? Consider first Figure 1, which corresponds to the case $f'(y_*) > 0$. Part (a) shows the graph of $f(y)$ for y near y_*; the sign of the first derivative of f tells us that $f(y)$ is negative to the left of y_* and positive to its right. From this the phase line shown in (b) and the qualitative sketches of solutions shown in (c) follow immediately; clearly y_* is a **source**. The case $f'(y_*) < 0$ is shown similarly in Figure 2; here y_* is a *sink*.

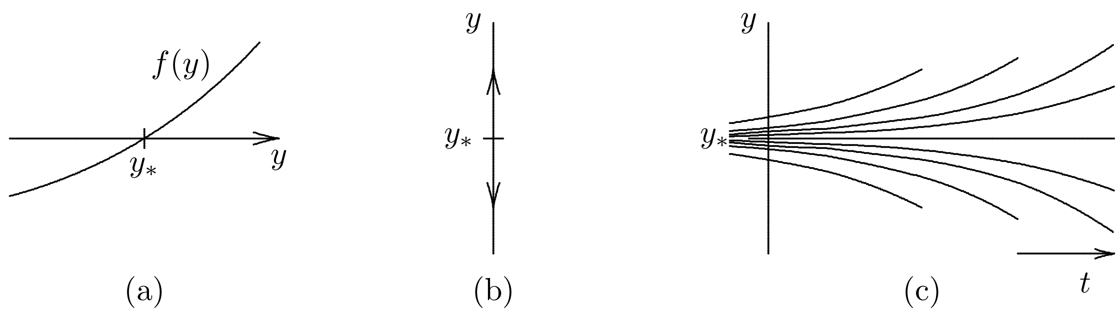

Figure 1: The case $f'(y_*) > 0$; here y_* is a source.
(a) The graph of $f(y)$ near y_*. (b) The phase line near y_*. (c) Solutions near y_*.

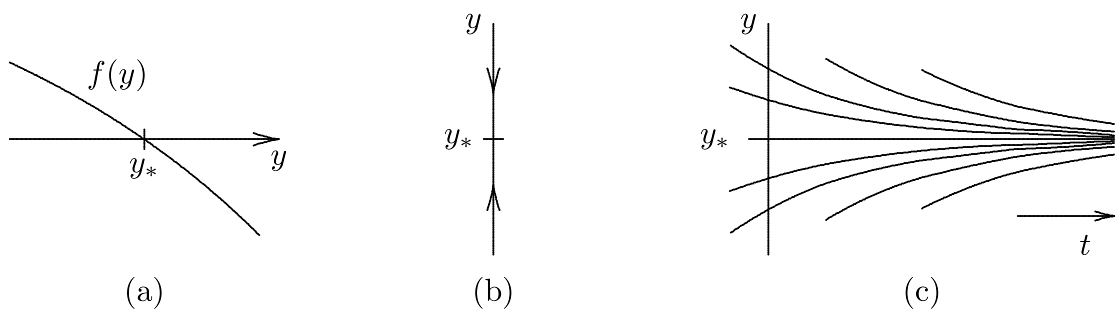

Figure 2: The case $f'(y_*) < 0$; here y_* is a sink.
(a) The graph of $f(y)$ near y_*. (b) The phase line near y_*. (c) Solutions near y_*.

Quantitative information from $f'(y_*)$. We can obtain analytic approximate solutions of the ODE near an equilibrium point through the process of *linearization*. Recall from elementary calculus the standard procedure for finding approximate values of a function $f(y)$ (which we assume to have as many derivatives as we need) near a point y_0. The most crude or *zeroth order* approximation is to approximate f by its value at y_0:

$$f(y) \approx f(y_0) \qquad \text{for} \quad y \approx y_0. \tag{2}$$

A better approximation is *first order* approximation:

$$f(y) \approx f(y_0) + f'(y_0)(y - y_0) \qquad \text{for} \quad y \approx y_0. \tag{3}$$

This is also called the *linear* approximation, because the approximation is linear in y, or the *tangent line* approximation, because the right hand side is the equation of the straight line tangent to the graph of $f(y)$ at the point $(y_0, f(y_0))$. One also consider higher order approximation of $f(y)$ by *Taylor polynomials*; the N^{th} approximation is

$$f(y) \approx \sum_{n=0}^{N} \frac{f^{(n)}(y_0)}{n!}(y - y_0)^n \qquad \text{for} \quad y \approx y_0. \tag{4}$$

If we take $N = 0$ or $N = 1$ in (4) we recover (2) or (3), respectively.

For our purposes here we deal only with (3); the idea is that for $y \approx y_*$ it is reasonable to replace the function $f(y)$ on the right hand side of the ODE (1) by its linear approximation near y_*, and use the fact that, since y_* is an equilibrium point, $f(y_*) = 0$. For ease of reading we will write $f'(y_*) = k$; then

$$\frac{dy}{dt} = f(y) \approx f(y_*) + f'(y_*)(y - y_0) = k(y - y_0), \qquad \text{for} \quad y \approx y_0. \tag{5}$$

To make the situation more transparent we introduce a new independent variable $u = y - y_*$; which measures the displacement of y from its equilibrium position; since $du/dt = dy/dt$, (5) becomes

$$\frac{du}{dt} \approx ku. \tag{6}$$

If we treat the approximation here as an equality then (6) becomes the simple linear ode $u' = ku$, with solution $u(t) = Ae^{kt}$ for some constant A. Returning to the variable y we obtain an approximate solution for the original equation (1):

$$y(t) \approx y_* + Ae^{kt} = y_* + Ae^{f'(y_*)t}. \tag{7}$$

Once again, this is a reasonable approximation only when $y \approx y_*$.

Suppose now that we have an initial condition $y(t_0) = y_0$ with $y_0 \approx y_*$. The approximate solution in (7) is then

$$y(t) = y_* + (y_0 - y_*)e^{kt}. \tag{8}$$

If $k = f'(y_*) < 0$ then the exponential function in (8) goes to zero as t goes to infinity; $y(t)$ gets closer to y_* and the linear approximation (3) used to derive (8) becomes better and better as time increases. In this case we may regard (8) as a good approximation for all $t \geq 0$. Obviously this is consistent with the discussion of the case $f'(y_*) < 0$ given in the previous section; see Figure 2. On the other hand, if $k = f'(y_*) > 0$ then the exponential function in (8) grows without bound as t goes to infinity; $y(t)$ moves away from y_* and the approximations we are using lose validity. Thus in this case (8) is worthless for large t. On the other hand, it is a good approximation for $t \leq 0$. See Figure 1.

In summary: For $f'(y_*) < 0$ the linearization analysis gives quantitative information for $t \geq 0$, predicting that solutions starting near y_* approach y_* exponentially fast—specifically with rate $|f'(y_*)|$. For $f'(y_*) > 0$ it gives only qualitative predictions for $t > 0$, but quantitative ones for $t \leq 0$.