Some comments on bifurcations

This is the definition of *bifurcate* from Merriam-Webster Dictionary: "to cause to divide into two branches or parts". (Etymology: Medieval Latin bifurcatus, past participle of bifurcare, from Latin bifurcus two-pronged, from bi- + furca fork.)

A quick way to look for bifurcation values of the parameter for the one-dimensional differential equation

$$\frac{dy}{dt} = f(y, \mu),$$

where $f(y, \mu)$ is a function for which both f and $\partial f / \partial y$ are continuous, begins by finding the simultaneous solutions of the system

$$f(y, \mu) = 0 \tag{1}$$

$$\frac{\partial f}{\partial y} = 0 \tag{2}$$

For each critical pair (y_0, μ_0), it is necessary to examine the behavior of the equation $y' = f(y(t), \mu)$ near the equilibrium point $y = y_0$ and try to determine whether bifurcation in fact occurs.

This procedure is very similar to the procedure which you followed in calc 1, when, in order to maximize a function, you first found critical points, and then studied, for each critical point, if it was a maximization point or not.

The solution of these equations may involve many technical difficulties, but examples in this course should allow all solutions to be found easily.

The justification of this procedure is that when the derivative on the left side of in (2) is **not** zero, one has, for any parameter near μ_0, some equilibrium near y_0 and of the same type (source or sink) and thus there is no change of behavior at that equilibrium point. The other equation, (1) just says that y is an equilibrium for this value of μ.

Here is another example, made up to show how solving two equations at the same time may be easier than considering them separately: if

$$f(y, \mu) = \sin y - \mu \cos y,$$

there are many possible equilibria for any given value of μ. (For example, when $\mu = 0$, all integer multiples of π are equilibria.) However, if we consider both equations (1)-(2) together, we have

$$\begin{aligned} \sin y - \mu \cos y &= 0 \\ \cos y + \mu \sin y &= 0 \end{aligned}$$

Multiplying the first of these by $\sin y$ and the second by $\cos y$ and adding yields $\sin^2 y + \cos^2 y = 0$. That is, if there are any solutions to this system of equations, we'd have $1 = 0$. The contradition shows that the system has no solutions.

In higher dimensional problems, the systematic nature of this approach becomes even more valuable.