# Exam 3 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam. It primarily (but not entirely) contains material covered after Exam 2. For a more thorough review of earlier material, see the corresponding review sheets (for Exams 1 and 2) and suggested homework in the textbook (on the Course Calendar).

**Note: For all problems asking for local phase portraits, make sure to include the correct tangencies at the equilibria from the straight-line solutions (if they exist), the direction of increasing time, and directions of rotation (if this makes sense for the specific linearization)**. Your phase portrait should include more than just one trajectory, and should give a general idea of the evolution of initial conditions near the equilibrium.

1. Consider the nonlinear system

$$\begin{aligned} x' &= x - 3y^2 \\ y' &= x - 3y - 6 \end{aligned}$$

   (a) Find all equilibria.

   (b) Classify the equilibria found in part (a), and draw a local phase portrait, **as accurately as possible**, near each equilibrium solution.

   (c) Using the information in (a) and (b), provide a rough sketch of the **global** phase portrait for the system. Note that this is just an approximation, but should be as accurate as possible.

2. Find the general solution to the forced second-order equation

$$\frac{d^2y}{dt^2} + ky = \cos\sqrt{k}t.$$

   Is this resonance? Plot the amplitude **of the forced response** as a function of time.

3. Consider the linear system $\mathbf{Y}' = A\mathbf{Y}$, where the matrix $A$ depends on a parameter $\alpha$ as below:

$$A = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix}.$$

   (a) As $\alpha$ is varied, a curve is traced out in the trace-determinant plane. Draw this curve, and indicate the direction of increasing $\alpha$.

   (b) Sketch phase portraits for the qualitatively different types of behavior exhibited by the system as $\alpha$ is varied.

   (c) Using your plots from (b), what are the bifurcation values of the system?

4. The following system has an equilibrium at $(\pi, \pi)$:

$$\begin{aligned} u' &= \sin(u - v) \\ v' &= \sin(u)\cos(v). \end{aligned}$$

   (a) Find the approximating linear system near $(\pi, \pi)$.

   (b) Sketch the trajectories of the phase portrait of the nonlinear system near $(\pi, \pi)$.

5. (a) Consider the second-order, constant-coefficient, homogeneous linear differential equation

$$ay'' + by' + cy = 0,$$

   where $a, b,$ and $c$ are **positive**. Show that all solutions of this equation tend to 0 as $t \to \infty$. Is this true if $b = 0$?

(b) Using the result of (a), show that all solutions of the corresponding **non-homogeneous** equation

$$ay'' + by' + cy = d,$$

where $d$ is a constant, tend to $\frac{d}{c}$ as $t \to \infty$. What happens if $c = 0$?

6. Solve the initial-value problem

$$\frac{dy}{dt} = (y+2)(y-1), \quad y(2) = 2.$$

7. Problem 23 in the Chapter 4 review problems (page 451 in the textbook).

8. For what values of $\alpha$ will the linear system

$$\mathbf{Y}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{Y}$$

have solutions which are spirals? Justify your answer.

9. This problem relates to material in Section 5.3. If curious for a more thorough treatment, please see that section in the textbook.

(a) Show that the system

$$\frac{dx}{dt} = -x + y + x^2$$
$$\frac{dy}{dt} = y - 2xy$$

can be written in the following form:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x},$$

for some function $H(x, y)$ (that is, that the system is **Hamiltonian**). To do this, note that we have the general property for mixed partial derivatives (as long as the function is "nice"):

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$$

Is this condition true, if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are given by the above system?

(b) Find the function $H(x, y)$. Note that this can be obtained via integration, in analogous ways one finds potential functions in Calculus III.

(c) Compute $\frac{d}{dt}H(x(t), y(t))$ along solution trajectories. You should use the chain rule, and the fact that you know what $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are.

(d) Use your result from (c) to find an **explicit** functional form for trajectories in the phase plane (i.e. $y = y(x)$). *Hint:* Use the quadratic formula.

10. Consider the system

$$\frac{dx}{dt} = -x^3 + xy^2$$
$$\frac{dy}{dt} = -2x^2 y - y^3.$$

Find a function $L(x, y)$ of the form $L(x, y) = ax^2 + by^2$ such that

(i) $L$ is non-negative ($L(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$), and

(ii) $L$ is decreasing along trajectories (i.e. $L$ is a Lyapunov function).

Note that the answer is not unique, and that you are asked to find **numbers** $a$ and $b$. Note that this is one type of method used to find a function mentioned at the end of Tuesday's (12/12) class *along which trajectories decrease.*

11. Solve the IVP

$$y' = \frac{x}{2y\sqrt{x^2-16}}, \quad y(5) = 2.$$

What is the largest interval to which the solution of the IVP exists?

12. When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.2 meters. (Gravitational acceleration is $9.8m/s^2$.) At $t = 0$, the mass is displaced $0.1m$ below its equilibrium (rest) position, and released with an initial upward velocity of $0.03m/s$. Assume that the spring force is proportional to its displacement, and that the mass is attached to a viscous-damper with a damping constant of $0.4N \cdot s/m$, and that the mass is acted on by an external force is $4\cos(2t)N$. Here $N$ denotes Newtons (don't worry, all the units work here).

(a) Formulate the above as a second-order differential equation, with initial conditions (i.e. a second-order IVP). *Hint:* Think $F = ma$.

(b) Convert your IVP in (a) to a first-order system. Don't forget to include initial conditions.

Note you do not need to solve anything here. I am just asking you to formulate the problem.

13. Consider the second-order non-homogeneous differential equation

$$y'' - \frac{1}{5}y' + 26y = \sin 5t. \tag{1}$$

**Using the method of complexification,** solve first $y'' - \frac{1}{5}y' + 26y = e^{i\omega t}$ (what is $\omega$ here?), in order to find a general solution of the original equation (1) in the form $y(t) = A\sin(\omega t + \phi)$. In other words, your final answer should be expressions for $A, \omega$, and $\phi$.

14. Consider the system of differential equations

$$x' = 1 - y^2$$
$$y' = x(2 - y).$$

This system (although not Hamiltonian) DOES possess a conserved quantity. Using the technique for finding an ODE for the curves in the phase plane (i.e. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, from Exam 1), find a separable differential equation for $y = y(x)$, and integrate it to find a conserved quantity $H(x,y)$. *Hint:* You may need to use synthetic division to integrate one of the sides.

15. Consider the following five first-order ODEs:

(i) $y' = 1 - y^2$

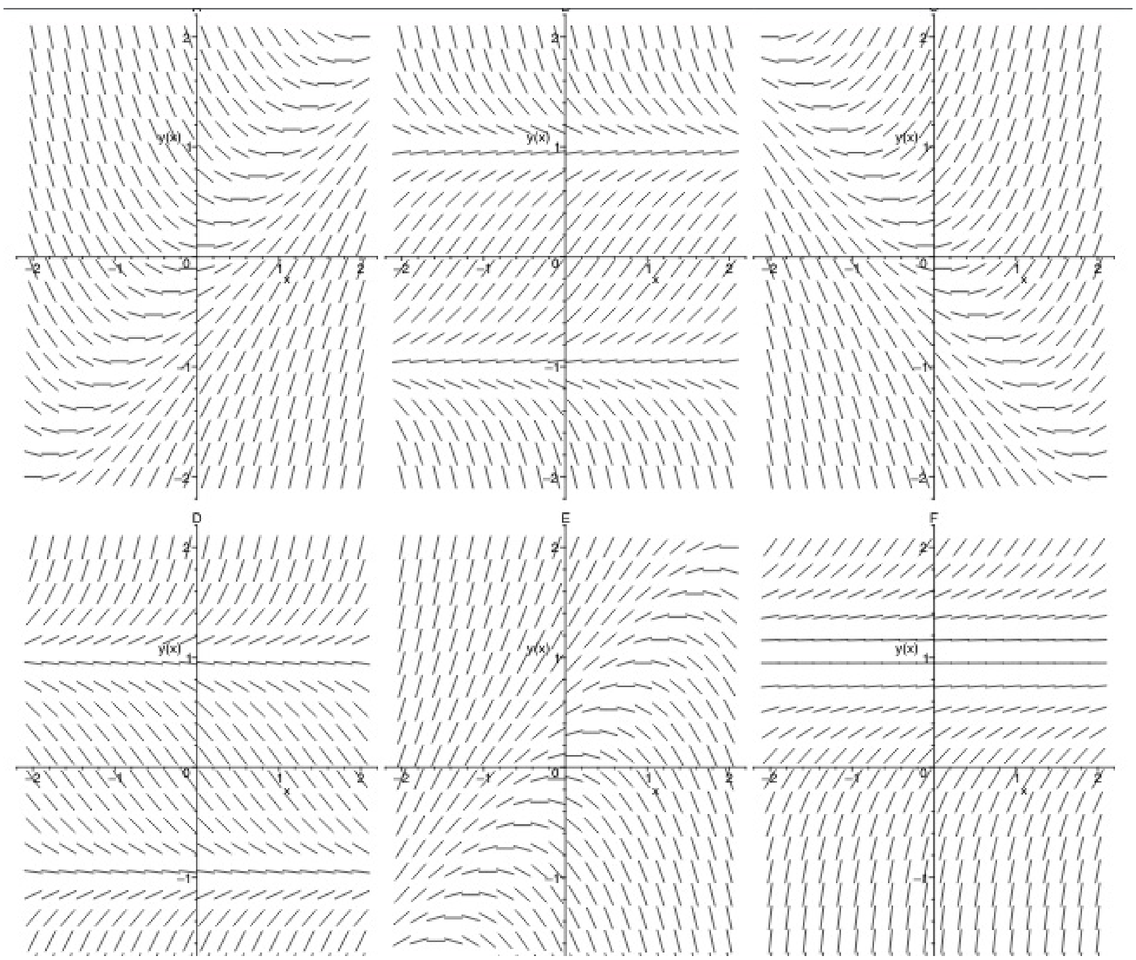
(ii) $y' = (1 - y)^2$

(iii) $y' = y - x$

(iv) $y' = x - y$

(v) $y' = x + y$

Below is a plot of direction (slope) fields. Match each of the above ODEs to its corresponding direction field.




16. For the autonomous ODE

$$y' = y(9 - y^2)(2 - y),$$

identify all equilibrium and their type (sink, source, node), and sketch all qualitatively distinct solution curves in the $(t, y)$ plane.

17. Suppose $B$ is a $2 \times 2$ matrix with the following eigenpairs:

$$\left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \qquad \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Find the general solution of the linear system

$$\mathbf{Y}' = B\mathbf{Y}.$$

Draw the corresponding phase portrait as well.