Exam 1 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

1. Find the solution of the initial-value problem

$$\begin{cases} \frac{dy}{dt} &= \frac{ty}{1+t^2} \\ y(1) &= 1. \end{cases}$$

2. A cup of soup is initially 170° F, and is left in a room with an ambient temperature of 70° F. Suppose that at time $t = 0$ it is cooling at a rate of 20° F per minute.

   (a) Assume **Newton's Law of Cooling** applies: The rate of cooling is proportional to the difference between the current temperature and the ambient temperature. Write down an IVP that models the temperature of the soup.

   (b) Solve the above IVP, for the temperature $T(t)$ at any time t.

   (c) How long does it take the soup to cool to a temperature of 110° F?

3. Consider the differential equation

$$\frac{dy}{dt} = y - t^2.$$

   Sketch a plot of the direction field for the above equation, and draw representative solution curves.

4. Problem #42, page 138 in the textbook (this is in the review exercises for Chapter 1).

5. Consider the ODE

$$y' = y^3 - 3y + a,$$

   where a is a parameter.

   (a) Using the graph of $y^3 - 3y$, find the values of the parameter a where any bifurcations occur. *Hint:* Note that the right-hand side is just vertical translates of this graph.

   (b) For all qualitatively distinct cases, draw the phase line for the system.

   (c) Sketch a bifurcation diagram for this equation.

6. The related problems #16 on page 50 and #11 on page 180 in the textbook.

7. How many equilibrium solutions does the system of differential equations

$$\begin{cases} \frac{dx}{dt} &= x(x-y) \\ \frac{dy}{dt} &= (x^2-4)(y^2-9) \end{cases}$$

   have? What are they?

8. Solve the following IVP

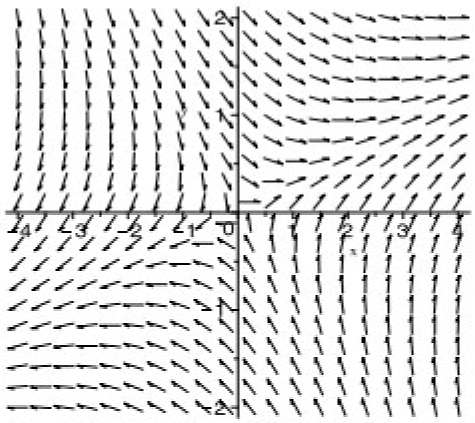
$$\begin{cases} \frac{dy}{dt} &= \frac{y}{t-2} + t, \\ y(3) &= -3. \end{cases}$$

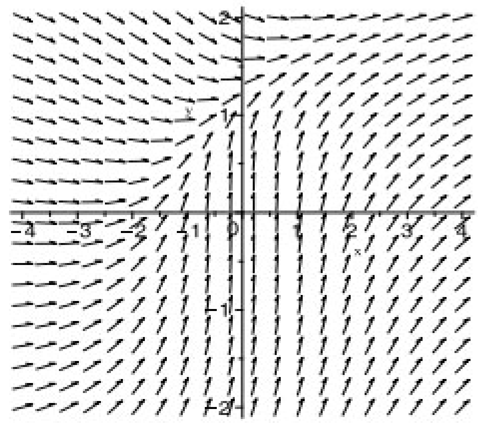
   What is the largest interval, containing the initial time, on which the solution is defined?

9. The figures below show two phase planes with direction fields. On each figure, sketch the trajectories (for both negative and positive times) satisfying the initial conditions

(a) $(x(0), y(0)) = (0, -1)$

(b) $(x(0), y(0)) = (-3, 1)$.





10. Convert the following second-order ODE into a first-order system:

$$t^2 y'' + ty' + (t^2 - 0.25)y = 0.$$

11. A 1000 gallon tank is initially full of a salt solution at a concentration of 1lb/gal. Pure water enters the tank at a rate of 1 gallon per minute and the well-stirred solution in the tank is siphoned out a rate of 3 gal/min. Find the concentration of salt in the tank when it is half full.

12. Find a solution of the initial-value problem (IVP)

$$\begin{cases} \frac{dy}{dx} &= \frac{y \cos x}{1+2y^2}, \\ y(0) &= 1. \end{cases}$$

Hint: Finding a solution does not necessarily mean finding an **explicit** solution $y = y(x)$. Solutions can often be thought of curves in the (here) xy-plane. Can you find an explicit solution to this equation?

13. Consider the second-order equation

$$y'' + \sin y = 0.$$

(a) Convert the above into a first-order system (i.e. $Y' = F(Y)$, where $Y \in \mathbb{R}^2$. You need to identify Y and $F(Y)$.).

(b) By hand, approximately draw the vector field obtained in part (a). Use this to then draw approximate solution trajectories in the phase plane. *Hint:* The second-order equation represents the motion of a free undamped pendulum.

14. Assume the volume $V(t)$ of a **cube** is increasing proportionally to the area of one of its faces.

(a) Write down an initial-value problem (IVP) for $V(t)$, assuming that the cube has an initial volume of 1 (arbitrary units). *Hint:* For a cube, $V = x^3$, where x is the side length. How can you relate x, and hence the area of a face, to the volume V?

(b) Solve the IVP obtained in part (a). Note that there will be an unknown constant in the formula.

(c) Assuming that $V(1) = 8$, find the constant, and give a precise formula for $V(t)$.