Systems Example

The analysis below should give a general idea of how a steady state linear analysis can be undertaken for a system. This analysis in requires some complicated algebra and calculus for arbitrary parameter values, and is probably more than I would ask on a HW or exam. However, I think it is important to be able to follow the general procedure, and understand why every step is being carried out, i.e. the "big picture" of the argument. Feel free to talk to me about this in office hours or by appointment if you have questions.

Here I recall the example mentioned in class of the predator-prey discrete system. Recall that the equations are given by

$$\begin{aligned} N_{t+1} &= rN_t e^{-aP_t} = f(N_t, P_t) \\ P_{t+1} &= N_t(1-e^{-aP_t}) = g(N_t, P_t) \end{aligned} \tag{1}$$

To find the steady states, we must find the set of points (N_*, P_*) such that

$$N_* = rN_* e^{-aP_*} \tag{2}$$

$$P_* = N_*(1-e^{-aP_*}) \tag{3}$$

The first equation (2) implies that $N_* = 0$ or $1 = re^{-aP}$, and thus plugging these values into equation (3) yields the steady states

$$\begin{aligned} (N_*, P_*) &= (0,0), \text{ and} \\ (N_*, P_*) &= \left(\frac{r}{a(r-1)}\log r, \frac{1}{a}\log r\right), \end{aligned} \tag{4}$$

Note that the second of the above only exists for $r > 1$.

To find the linear stability of the 2 steady states in (4), we need to compute the Jacobian matrix A and evaluate it at the steady states. Using basic calculus, you can see that

$$A(N,P) = \begin{pmatrix} \frac{\partial f}{\partial N}(N,P) & \frac{\partial f}{\partial P}(N,P) \\ \frac{\partial g}{\partial N}(N,P) & \frac{\partial g}{\partial P}(N,P) \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} re^{-aP} & -arNe^{-aP} \\ 1-e^{-aP} & aNe^{-aP} \end{pmatrix} \tag{6}$$

Substituting the steady states in the above yields matrices

$$A(0,0) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

and

$$A\left(\frac{r}{a(r-1)}\log r, \frac{1}{a}\log r\right) = \begin{pmatrix} 1 & -\frac{r}{r-1}\log r \\ 1-\frac{1}{r} & \frac{1}{r-1}\log r \end{pmatrix}. \tag{8}$$

Now, our main theorem says that the steady state $(\mathbf{N}_*, \mathbf{P}_*)$ **is stable if and only if all of the eignevalues of the matrix** $\mathbf{A}(\mathbf{N}_*, \mathbf{P}_*)$ **have magnitude less than 1.** $(0,0)$ is easy, since from (7) we see that it has eigenvalues $\lambda_1 = r, \lambda_2 = 0$, and hence is stable when $r < 1$ and unstable when

$r > 1$ (remember we only consider $r > 0$ in the model). The second fixed point is less trivial, since the matrix (8) is non-diagonal. The characteristic equation (the equation whose zeros are the eigenvalues) of this matrix is, after a bit of algebra

$$\lambda^2 - \left(1 + \frac{1}{r-1}\log r\right)\lambda + \frac{r}{r-1}\log r = 0, \tag{9}$$

which has zeros (i.e. eigenvalues)

$$\lambda_{1,2} = \frac{1}{2}\left(1 + \frac{\log r}{r-1} \pm \sqrt{\left(1 + \frac{\log r}{r-1}\right)^2 - 4\frac{r\log r}{r-1}}\right). \tag{10}$$

A little algebra and calculus can be used to show that what is under the square root above is negative for $r > 1$ (which is the only case where this fixed point exists), so that the roots are complex conjugates, and thus have the same magnitude. And, as the product of the roots is always the constant term in a polynomial, the roots have common magnitude

$$\lambda_1\lambda_2 = |\lambda|^2 = \frac{r}{r-1}\log r > 1 \tag{11}$$

where the inequality can be shown again using calculus, for $r > 1$. Thus, the second steady state is **always unstable when it exists, since the corresponding Jacobian has at least one (in fact both) eigenvalues with magnitude greater than 1.**

Why is the theorem true?

I wanted to give a brief intuitive reason for the role the eigenvalues play in determining linear stability. Remember, we want to solve the linear system for the perturbations, which is given by

$$\begin{pmatrix}\tilde{n}_{t+1}\\ \tilde{p}_{t+1}\end{pmatrix} = A\begin{pmatrix}\tilde{n}_t\\ \tilde{p}_t\end{pmatrix}, \tag{12}$$

where A is the Jacobian matrix evaluated at the steady state, and that $\begin{pmatrix}\tilde{n}_t\\ \tilde{p}_t\end{pmatrix} \to \begin{pmatrix}0\\ 0\end{pmatrix}$ is equivalent to the steady state being stable.

We investigate by looking for solutions of the form

$$\begin{pmatrix}\tilde{n}_t\\ \tilde{p}_t\end{pmatrix} = \mathbf{v}\lambda^t \tag{13}$$

where $\mathbf{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ (and neither of which are zero, since we don't start at the steady state). Plugging this expression into (12), and simplifying yields

$$\mathbf{v}\lambda^{t+1} = A\mathbf{v}\lambda^t, \quad \text{or} \tag{14}$$

$$(A - \lambda I)\mathbf{v} = 0 \tag{15}$$

But this is precisely the eignevalue-eigenvector equation. That is, λ must be an eigenvalue of A, with corresponding eigenvector $\mathbf{v}$. Since the equation is linear, the general solution then can be written as

$$\begin{pmatrix}\tilde{n}_t\\ \tilde{p}_t\end{pmatrix} = c_1\mathbf{v}_1\lambda_1^t + c_2\mathbf{v}_2\lambda_2^t, \tag{16}$$

for arbitrary constants c_1, c_2 determined by the initial perturbation. In general we don't care about the exact solution (it is a linear approximation anyway), but just the limit as $t \to \infty$. As in the one-dimensional case, the stability is precisely determined by the magnitude of λ_1, λ_2, as stated in the theorem. That is, the perturbations shrinks to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$ if and only if $|\lambda_1|, |\lambda_2| < 1$. Note lastly that I am ignoring some degenerate cases, such as the case of identical eigenvalues. These can be handled in similar ways (which I won't discuss), but is exactly like repeated roots in your ODE course.