Exam 3 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam. It primarily (but not entirely) contains material covered after Exam 2. For a more thorough review of earlier material, see the corresponding review sheets (for Exams 1 and 2) and suggested homework in the textbook (on the Course Calendar).

Note: For all problems asking for local phase portraits, make sure to include the correct tangencies at the equilibria from the straight-line solutions (if they exist), the direction of increasing time, and directions of rotation (if this makes sense for the specific linearization). Your phase portrait should include more than just one trajectory, and should give a general idea of the evolution of initial conditions near the equilibrium.

1. Consider the nonlinear system

$$\begin{aligned} x' &= x - 3y^2 \\ y' &= x - 3y - 6 \end{aligned}$$

   (a) Find and classify all equilibria. Using this information, draw a local phase portrait, **as accurately as possible**, near each equilibrium solution.
   (b) Sketch the nullclines of the system. Include the general direction of motion of trajectories (i.e. the vector field) in all regions of phase space.
   (c) Using the information in (a) and (b), provide a rough sketch of the **global** phase portrait for the system.

2. Find the general solution to the forced second-order equation

$$\frac{d^2y}{dt^2} + ky = \cos \sqrt{k}t.$$

   Is this resonance? Plot the amplitude **of the forced response** as a function of time.

3. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x^2 - 3y^2 \\ \frac{dy}{dt} &= 1 - y. \end{aligned}$$

   (a) Find and classify all equilibria. Using this information, draw a local phase portrait, **as accurately as possible**, near each equilibrium solution.
   (b) Sketch the nullclines of the system. Include the general direction of motion of trajectories (i.e. the vector field) in all regions of phase space.
   (c) Using the information in (a) and (b), provide a rough sketch of the **global** phase portrait for the system.

4. The following system has an equilibrium at (π, π):

$$\begin{aligned} u' &= \sin(u - v) \\ v' &= \sin(u)\cos(v). \end{aligned}$$

   (a) Find the approximating linear system near (π, π).
   (b) Sketch the trajectories of the phase portrait of the nonlinear system near (π, π).

5. Solve the initial-value problem

$$\frac{dy}{dt} = (y + 2)(y - 1), \quad y(2) = 2.$$

6. Problem 23 in the Chapter 4 review problems (page 451 in the textbook).

7. For what values of α will the linear system

$$\mathbf{Y}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{Y}$$

have solutions which are spirals? Justify your answer.

8. Show that the system

$$\frac{dx}{dt} = -x + y + x^2$$
$$\frac{dy}{dt} = y - 2xy$$

is Hamiltonian, and find a corresponding Hamiltonian function $H(x, y)$. Using $H(y, x)$, find an **explicit** functional form for trajectories in the phase plane (i.e. $y = y(x)$). *Hint:* Use the quadratic formula.

9. Consider the system

$$\frac{dx}{dt} = -x^3 + xy^2$$
$$\frac{dy}{dt} = -2x^2y - y^3.$$

Find a function $L(x, y)$ of the form $L(x, y) = ax^2 + by^2$ such that

(i) L is non-negative ($L(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$), and

(ii) L is decreasing along trajectories (i.e. L is a Lyapunov function).

Note that the answer is not unique, and that you are asked to find **numbers** a and b.

10. Solve the IVP

$$y' = \frac{x}{2y\sqrt{x^2 - 16}}, \quad y(5) = 2.$$

What is the largest interval to which the solution of the IVP exists?

11. When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.2 meters. (Gravitational acceleration is $9.8 m/s^2$.) At $t = 0$, the mass is displaced $0.1m$ below its equilibrium (rest) position, and released with an initial upward velocity of $0.03 m/s$. Assume that the spring force is proportional to its displacement, and that the mass is attached to a viscous-damper with a damping constant of $0.4 N \cdot s/m$, and that the mass is acted on by an external force is $4\cos(2t)N$. Here N denotes Newtons (don't worry, all the units work here).

(a) Formulate the above as a second-order differential equation, with initial conditions (i.e. a second-order IVP). *Hint:* Think $F = ma$.

(b) Convert your IVP in (a) to a first-order system. Don't forget to include initial conditions.

Note you do not need to solve anything here. I am just asking you to formulate the problem.

12. Consider the second-order non-homogeneous differential equation

$$y'' - \frac{1}{5}y' + 26y = \sin 5t. \tag{1}$$

Using the method of complexification, solve first $y'' - \frac{1}{5}y' + 26y = e^{i\omega t}$ (what is ω here?), in order to find a general solution of the original equation (1) in the form $y(t) = A\sin(\omega t + \phi)$. In other words, your final answer should be expressions for A, ω, and ϕ.

13. Consider the system of differential equations

$$x' = 1 - y^2$$
$$y' = x(2 - y).$$

(a) Show that the above system is NOT Hamiltonian.

(b) However, I claim that the system DOES possess a conserved quantity. Using the technique for finding an ODE for the curves in the phase plane (i.e. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, from Exam 1), find a separable differential equation for $y = y(x)$, and integrate it to find a conserved quantity $H(x, y)$. *Hint:* You may need to use synthetic division to integrate one of the sides.

14. Consider the following five first-order ODEs:

(i) $y' = 1 - y^2$

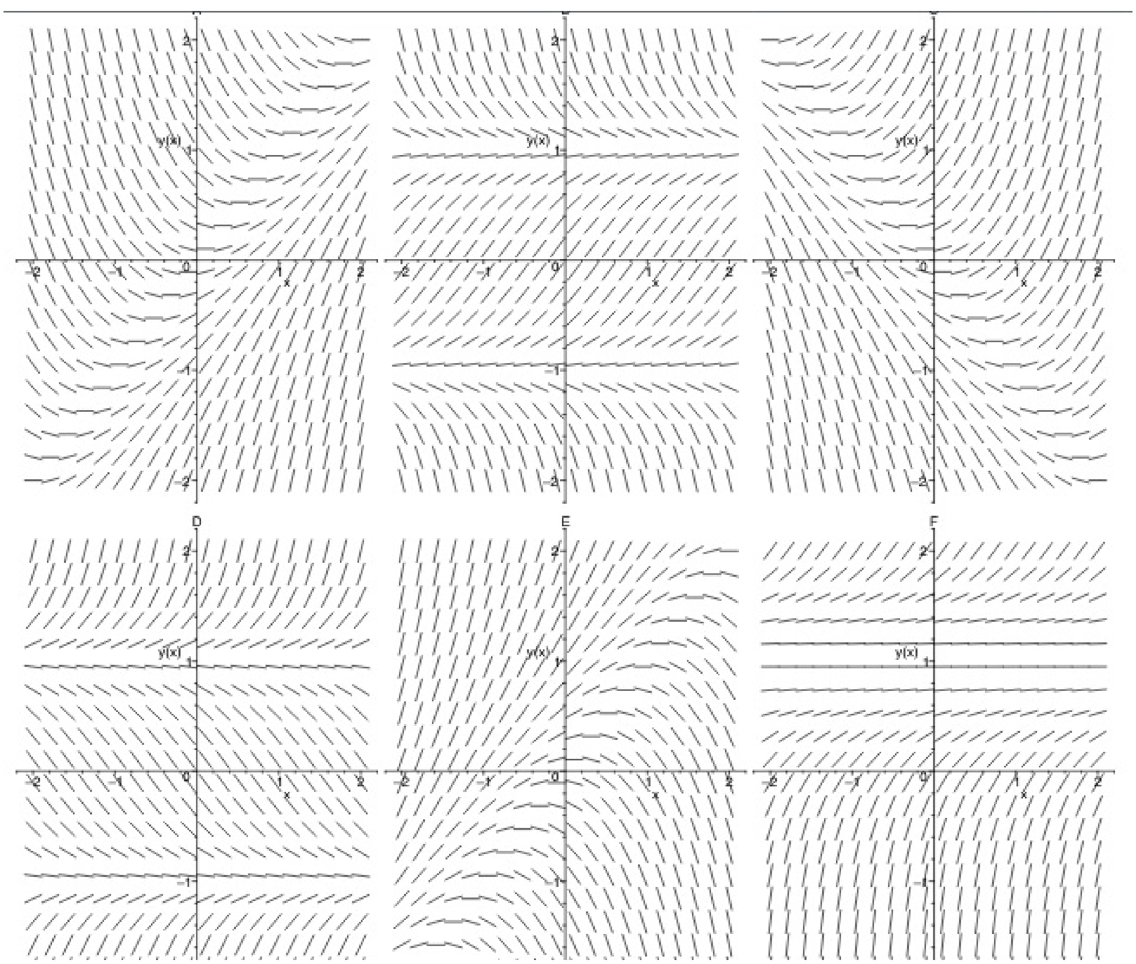
(ii) $y' = (1 - y)^2$

(iii) $y' = y - x$

(iv) $y' = x - y$

(v) $y' = x + y$

Below is a plot of direction (slope) fields. Match each of the above ODEs to its corresponding direction field.



15. For the autonomous ODE

$$y' = y(9 - y^2)(2 - y),$$

identify all equilibrium and their type (sink, source, node), and sketch all qualitatively distinct solution curves in the (t, y) plane.

16. Suppose B is a 2×2 matrix with the following eigenpairs:

$$\left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \qquad \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Find the general solution of the linear system

$$\mathbf{Y}' = B\mathbf{Y}.$$

Draw the corresponding phase portrait as well.