Exam 2 Review Problems

For details on exam coverage and a list of topics, please see the class website. Note that this is a set of review problems, and NOT a practice exam.

Note: For all problems asking for phase portraits, make sure to include straight-line solutions (if they exist), correct tangents as t $\to \pm\infty$, the direction of increasing time, and directions of rotation (if this makes sense for the specific matrix). Your phase portrait should include more than just one trajectory, and should give a general idea of the evolution of ANY initial condition.

1. (a) Find the general solution of the system

$$\mathbf{Y}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{Y}.$$

   (b) Draw the phase portrait of the above system.

   (c) Use the general solution from part (b) to solve the IVP $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   (d) Using the phase portrait and/or the solution found in (c), plot the components $x(t)$ and $y(t)$ (separate or same set of axes, your choice).

2. Compute the equilibrium points for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

3. Suppose that $\mathbf{Y}_1(t) = (-e^{-t}, e^{-t})$ is a solution to some linear system $d\mathbf{Y}/dt = A\mathbf{Y}$. If possible, find an **explicit** solution to the the same system with the following initial conditions. If not possible, state why. *Note:* Solutions exist for all initial conditions, but here I am asking you to find a formula, if possible.

   (a) $\mathbf{Y}(0) = (-2, 2)$

   (b) $\mathbf{Y}(0) = (3, 4)$

   (c) $\mathbf{Y}(0) = (0, 0)$

   (d) $\mathbf{Y}(0) = (3, -3)$

4. Consider the linear system

$$\mathbf{Y}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{Y}.$$

   (a) Find the general solution for the above system.

   (b) Draw the corresponding phase portrait.

   (c) Use the general solution from part (a) to solve the initial-value problem $\mathbf{Y}(0) = (3, 2)$.

   (d) Plot the first component, $x(t)$ of the solution in part (c) as a function of time ($t \in (-\infty, \infty)$). Be sure to include all pertinent information (horizontal asymptotes, relative maxima/minima, etc.).

5. (a) Consider the second-order, constant-coefficient, homogeneous linear differential equation

$$ay'' + by' + cy = 0,$$

   where a, b, and c are **positive**. Show that all solutions of this equation tend to 0 as $t \to \infty$. Is this true if $b = 0$?

(b) Using the result of (a), show that all solutions of the corresponding **non-homogeneous** equation

$$ay'' + by' + cy = d,$$

where d is a constant, tend to $\frac{d}{c}$ as $t \to \infty$. What happens if $c = 0$?

6. Problem #21, page 379 in the textbook (this is in the review exercises for Chapter 3).

7. (a) Compute the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

(b) Sketch the phase portrait.

(c) Solve the IVP, with $\mathbf{Y}(0) = (-2, 3)$.

(d) Sketch the $x(t)$ and $y(t)$ graphs for the solution of the initial-value problem found in part (c).

8. In the absence of damping, the motion of a spring-mass system satisfies the initial-value problem

$$my'' + ky = 0, \quad y(0) = a, \quad y'(0) = b.$$

(a) Show that the kinetic energy ($\frac{1}{2}mv^2$) *initially* imparted to the mass is $mb^2/2$, and that the potential energy *initially* stored in the spring is $ka^2/2$, so that the total energy initially is $(ka^2 + mb^2)/2$.

(b) Solve the IVP.

(c) Using the solution from part (b), determine the total energy in the system *at any time* t. How does this relate to a well-known physical principle?

9. Consider the linear system $\mathbf{Y}' = A\mathbf{Y}$, where the matrix A depends on a parameter α as below:

$$A = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix}.$$

(a) As α is varied, a curve is traced out in the trace-determinant plane. Draw this curve, and indicate the direction of increasing α.

(b) Sketch phase portraits for the qualitatively different types of behavior exhibited by the system as α is varied.

(c) Using your plots from (b), what are the bifurcation values of the system?

10. Sketch the phase portrait for the system

$$\mathbf{Y}' = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

11. Consider the matrix $A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$.

(a) Find all eigenvalues and a corresponding eigenvector for A.

(b) Find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

(c) Compute e^{At} using your result from part (b). Simplify to obtain a single 2×2 matrix.

(d) Solve the IVP

$$\mathbf{Y}' = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{Y},$$
$$\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

12. Consider the second-order, non-homogeneous equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 8t + 2.$$

(a) Find a general solution of the corresponding homogeneous equation, $y_h(t)$.

(b) Find a particular solution, $y_p(t)$, of the non-homogeneous equation.

(c) Find the general solution of the non-homogeneous equation.

(d) How would you describe the behavior of the general solution found in part (c) for large time t? Provide an approximate plot of what the solution looks like, after an initial transient amount of time has elapsed. *Hint:* This should be **independent** of initial conditions (i.e. arbitrary constants).