Matlab Assignment #7

The Piano and the Violin

Refer to the file *ExampleSound.m* for this Matlab assignment. Your answers should be accurate to 3 significant figures. For any Matlab plot you submit, use the function *title* to give the plot a title name which identifies which problem number. For questions which ask you to play sounds with Matlab, you *do not* need to turn in any sound files.

Sound travels through the air as compression waves. In music, the note A above middle C is usually tuned to the frequency to 440 Hz. A *Hertz* is one cycle per second, and is abbreviated as Hz. To designate octaves, these notes are referred to as A_4 and C_4. The note A_0 is the lowest key on the piano, and C_8 is the highest key on the piano.

One note is said to be an *octave* above the other if its frequency is twice that of the first. For example, the note A_5 is one octave above A_4 and its frequency is $2 \times 440 = 880$ Hz. The note A_6 is two octaves above A_4 and its frequency is $2^2 \times 440 = 1760$ Hz.

One note is said to be a *perfect fifth* above the other if its frequency is $\frac{3}{2}$ times that of the first. The note E_5 is a perfect fifth above A_4 and its frequency is $\frac{3}{2} \times 440 = 660$ Hz. Also, the note B_5 is two perfect fifths above A_4, and its frequency is $(\frac{3}{2})^2 \times 440 = 990$ Hz.

There are 12 notes, or semitones, in an octave, and there are 7 semitones in a perfect fifth. *Pythagorean tuning* determines the frequency of each note in the scale using only the ratios of 2:1 from octaves and 3:2 from perfect fifths.

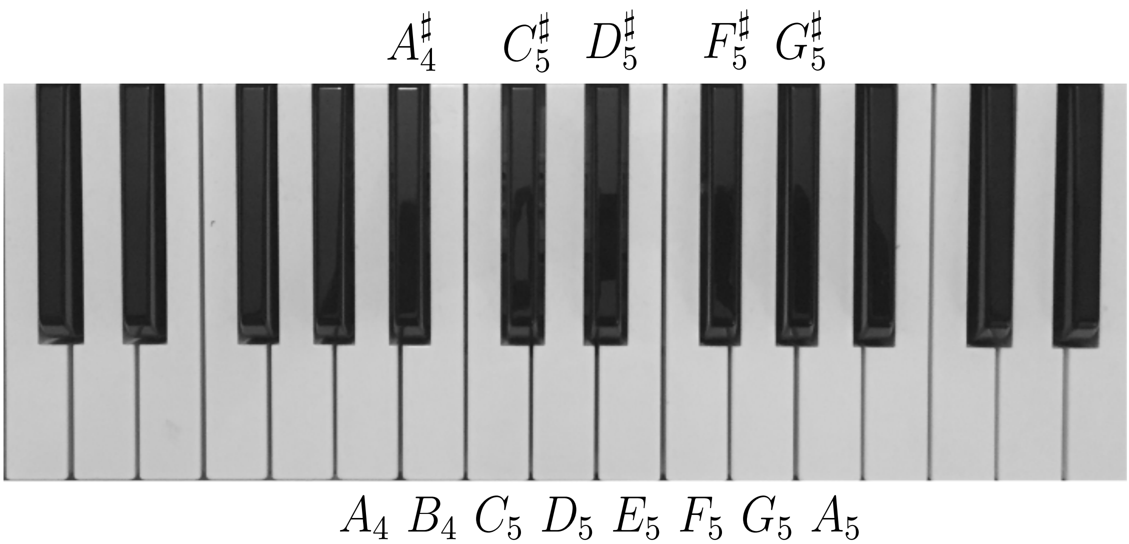

Figure 1: The keys of a piano

1. Unfortunately, the Pythagorean method of tuning runs into some problems. Compute the frequency ω_1 of the note A_7 from being 7 octaves above A_0. (Hint: Since the frequency of A_4 is 2^4 times the frequency of A_0, it follows that the frequency of A_0 is $440/16 = 27.5$ Hz. Now compute the frequency of A_7.)

2. Use the Pythagorean method of tuning to compute the frequency ω_2 of A_7 from being 12 perfect fifths above A_0. (Hint: Your answers in questions #1 and #2 should NOT match. They will be off by about 1.36%)

3. If the two notes A_7 from questions #1 and #2 were played at the same time, their combined sound could be modeled by the function $f(t) := \sin(2\pi\omega_1 t) + \sin(2\pi\omega_2 t)$. For the function $f(t)$:

(a) Calculate the beating frequency of the function $f(t)$ (Hint: For calculating beating frequency, see Section 4.3 in the book

(b) Plot the function over the time $0 \leq t \leq 1$.

(c) Use *soundsc* to listen to the function $f(t)$.

In order to correct for this problem, other methods of tuning were invented, such as Ptolemaic tuning, and the meantone temperament. The *equal temperament* system of tuning was independently invented in China and in Europe during the 16^{th} century, and is commonly used today.

In equal temperament, the frequencies of the 12 notes in an octave are equally spaced – in a logarithmic sense. That is, there is a fixed constant γ such that a note N semitones above A_4 has a frequency of $\gamma^N \times 440$ Hz. Since A_5 is both an octave above A_4 and 12 semitones above A_4, then in order for the frequency of A_5 to be twice that of A_4, this forces $\gamma^{12} \times 440 = 2 \times 440$, or $\gamma = \sqrt[12]{2}$.

While modern pianos use equal temperament, violins often use Pythagorean tuning. Unless corrected for this can cause a small amount of dissonance.

4. Suppose a pianist and a violinist both play the note E_5. After the piano hits its strings, the note begins to fade, however the violinist continues to bow her string. The violin produces a sound with frequency is $\frac{3}{2} \times 440\ Hz = 660\ Hz$. The natural frequency of the piano's string is $(2^{1/12})^7 \times 440$ Hz. The position of the pianist's string is approximately given by the following differential equation:

$$y'' + 0.35196\ y' + 17158003.5\ y = \sin(2\pi\ 660\ t).$$

(Remark: While the constants above are not entirely physical, the constants were chosen so that the system would be underdamped and the natural frequency would be close to $(2^{1/12})^7 \times 440$ Hz.)

(a) Find the general solution to the problem.

(b) Solve the IVP with $y(0) = 10^{-4}$ and $y'(0) = 0$.

(c) Graph your solution over the range $0 \leq t \leq 20$.

(d) Listen to your solution using *soundsc*.

5. The note $C_5^\sharp$ is a major third above A_4, or plus four perfect fifths and minus two octaves. In Pythagorean tuning, if A_4 is 440 Hz, then $C_5^\sharp$ has a frequency of $\omega_1 = (3/2)^4 \times (1/2)^2 \times 440$. Since $C_5^\sharp$ is 4 semi-tones above A_4, then in equal temperament, $C_5^\sharp$ has a frequency of $\omega_2 = (2^{1/12})^4 \times 440$ Hz.

Suppose that a pianist and a violinist both play the note $C_5^\sharp$ at the same time. For simplicity, we can suppose that the piano string does not have any damping. The piano string can then be modeled by the following differential equation:

$$y'' + (2\pi\omega_2)^2\, y = \sin(2\pi\omega_1 t).$$

(a) Find the general solution to the problem

(b) Solve the IVP with $y(0) = 10^{-5}$ and $y'(0) = 0$.

(c) Calculate the beating frequency.

(d) Graph your solution over the range $0 \le t \le 2$.

(e) Listen to your solution using *soundsc*.