GRE MATHEMATICS TEST VI

DETAILED EXPLANATIONS OF ANSWERS

1. (C)

Note that the limit takes on the indeterminate form 1^∞. Thus, we apply L'Hopital's rule several times to the logarithm of $f(x)$:

$$\ln f(x) = \frac{\ln \frac{\sin x}{x}}{x^2} \sim \frac{x\cos x - \sin x}{2x^2 \sin x} \sim -\frac{\sin x}{4\sin x + 2x\cos x}$$

$$\sim -\frac{\cos x}{6\cos x - 2x\sin x} \sim -\frac{1}{6}.$$

Thus, $\ln f(x) \to -1/6$ or $f(x) \to -e^{-1/6}$.

2. (B)

A direct calculation shows that $\vec{a} \cdot \vec{c} = 0$, which implies that $\vec{a}$ and $\vec{c}$ are orthogonal. The other answers are all false.

3. (C)

Note that the intervals are nested, i.e., $A_{n+1} \subset A_n$. The limiting interval is therefore $(1, 2)$. However, the endpoints of this interval are
[illegible] every A ... Consequently, the [illegible]

4. (A)

By partial fraction decomposition,

$$\frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

The partial sums of the series form a telescoping sum, so that

$$\sum_{n=1}^{m} \frac{1}{4n^2-1} = \frac{1}{2}\left[(-1+1) + \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{2m-1} - \frac{1}{2m+1}\right)\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2m+1}\right).$$

and the partial sums $\to \frac{1}{2}$ as $m \to \infty$.

5. (E)

Applying the ratio test, it can be shown that

$$\left|\frac{a_{n+1}}{a_n}\right| \to e^{-\sin x}.$$

Convergence therefore follows when $\sin x > 0$. This holds whenever $2n\pi < x < (2n+1)\pi$ for any integer n. At the endpoints of these intervals, $\sin x = 0$, and each term of the series is $1/n^2$. In that case, the series also converges. Thus, the closed intervals given by (E) represent the allowable values for x.

6. (D)

Note that $V_5 = \{1, 2, 3, 4\}$. Since $1^4 = 1$, $2^4 = 16 = \bar{1}$, $4^2 = 16 = \bar{1}$, and $3^2 = 9 = \bar{4}$, it follows that every element of V_5 is a solution to the given equation.

7. (E)

Note first that $-if(z) = g(z)$. Since $f(z)$ is analytic, so is $g(z)$; thus I is true. However, $f'(z) = u_x + iv_x$, and so II is false. Finally, since $v(x, y)$ is harmonic, it satisfies Laplace's equation. The second partials of $(x + y)$ vanish, and so III is true.

8. (B)

Taking logarithmic derivatives, we find that $f'(x) = x^x(\ln x + 1)$. This vanishes when $x = 1/e$. Since $f(x) \to 1$ as $x \to 0$ or 1, the minimum of $f(x)$ occurs at $1/e$.

9. (D)

The indicated mapping is a linear fractional transformation which always maps circles (and straight lines) into circles and straight lines. The ellipse (III) and hyperbola (V) cannot be images of a circle under a linear fractional transformation.

10. (C)

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This matrix has eigenvalues $\{0, 1\}$ and eigenvectors $(1, 0)^T$ and $(0, 1)^T$. These eigenvectors are clearly linearly independent. Thus, I is *sometimes* true. By the Cayley-Hamilton theorem, every square matrix A satisfies its characteristic polynomial. Thus $p(A)$ is the zero matrix and II is *always* true. Finally, if $\lambda = 0$ is an eigenvalue of A, the $\det(A - \lambda I) = \det(A) = 0$, which implies that A is singular and therefore not invertible. III is *never* true.

11. (A)

Rewrite the equation as

$$e^{3z} = -i = e^{i(-\pi/2+2n\pi)}\text{ , } n \text{ an integer,}$$

which implies that

$$z = (i/3)(\pi/2 + 2n\pi).$$

12. (C)

Observe that

$$\frac{1}{z-2} = \frac{1}{1+z-3} = \frac{1}{z-3}\,\frac{1}{1-\dfrac{-1}{z-3}},$$

and expanding this last term in a geometric series yields the Laurent series given by (C).

13. (E)

The four petals in the figure indicate that the equation must contain an argument proportional to 4θ. Of the two possibilities, only (E) intersects the point (2, 0), which is on the x–axis.

14. (E)

The algorithm described in Newton's method whose correct formula is given by (E).

15. (D)

$P(A)$ contains $2^3 = 8$ elements and B contains 2 elements, so that

C contains 16 elements. $P(C)$ therefore contains 2^{16} elements.

16. (D)

The differential equation is of the Cauchy-Euler type. The indicial equation is found by making the substitution $y = x^r$ which yields

$$r(r-1)\,x^2x^{r-2} + 4rxx^{r-1} + 4x^r = 0.$$

Then we cancel out the common factor x^r from each term.

7. (A)

The non-invertible elements are those whose squares are equal to he additive identity. Since $3^2 = 9 = 0 \bmod 9$ and $6^2 = 36 = 0 \bmod 9$, the orrect answer is (A).

8. (E)

There are 9 possibilities for the second digit and 8 possibilities for e third digit. In addition, there are 25 possibilities for the double set f letters. Thus, there are $9 \cdot 8 \cdot 25 = 1800$ possible license plates which atch the stated description.

. (A)

Let T be the temperature and t be time. From the problem atement, the rate of change of temperature, which is the derivative $/dt$ is proportional to the difference between the object's temperature and the ambient temperature which is zero. Thus, $dT/dt = -kT$, where $\cdot 0$ (the object must cool–thus the negative sign). The solution to this fferential equation is $T = T_0e^{-kt}$ where T_0 is the initial temperature. hen $t = 1$, $T = T_0/2$, or $T_0/2 = T_0e^{-k}$. Solving for k, we obtain $k = \ln 2$.

20. (C)

An element g is of order 2 if $g^2 = e$ but $g \neq e$. Since e is one of the m elements satisfying the indicated relation, there must be exactly $m - 1$ elements of order 2 in the group.

21. (B)

The function $f(x)$ has a unique fixed point when it maps $[a, b]$ into $[a, b]$ and $|f'(x)| \leq k \leq 1$. The functions in I and II satisfy these conditions. However, III violates the bound on the derivative.

22. (D)

A tautology is a logical form which is always true, regardless of the truth value of its components. In II, $(P \wedge \neg P)$ is always false. Theorems with a false hypothesis are always true, independent of the truth value of the conclusion. In the other hand, in I, if P and Q are both false, the statement is false, while if P and Q are both true, the statement is true. Thus I cannot be a tautology. Similarly, in III, if P and Q are both false, the statement is true, while if P and Q are both true, the statement is false. Thus, II is the only tautology.

23. (B)

A linear functional in R^3 must have the form $f(\alpha) = ax + by + cz$ where $\alpha = (a, b, c)$ and x, y, and z are real constants. Using the information given in the problem statement, a system of three equations for x, y, and z are obtained, with solution $x = 4$, $y = -7$, and $z = -3$. Thus, the correct form of the functional is given by (B).

24. (E)

The three elements of U form an abelian group under addition. The only possible value for $1 + 1$ is c, so II holds. By the same token, $c + 1 = 0$ and, using II, I is also true. Finally, by constructing the multiplicative table for U, $c^2 = c \cdot c = 1$ must also hold. Thus, all three statements are true.

25. (A)

The cross product of any two (non-parallel) vectors in the plane must be normal to the plane. Two such vectors are those joining the given points, namely $\vec{v} = (3, -2, 1)$ and $\vec{w} = (-2, -2, -2)$. Thus, $\vec{v} \times \vec{w} = (8, 4, -12)$. If $P_0 = (x_0, y_0, z_0)$ is a point in the plane, then the equation of the plane is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\,,$$

where (A, B, C) is normal to the plane. Using our computed results for the normal, and any of the three given points for P_0 we obtain (A) as the equation for the plane.

26. (E)

Since $1 \mapsto 3 \mapsto 2 \mapsto 1$ and $5 \mapsto 4 \mapsto 5$, it is straightforward to show that $\beta^7 = \beta$. Thus β^6 is the identity, and so the order of β is six.

27. (D)

The characteristic polynomial of A is found from the equation $|A - \lambda I| = 0$:

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0\,.$$

This determinant evaluates to the polynomial $\lambda^2 - 2\lambda - 2 = 0$. This polynomial also annihilates A, i.e., $A^2 - 2A - 2I = 0$.

28. (D)

First, note that $(a \circ b) - (b \circ a) = b - a$, so the operation is not commutative. Further, $[a \circ (b \circ c)] - [(a \circ b) \circ c] = 2c - ac$. Thus the operator is not associative. Finally, it is easy to show that inverses in general do not exist. Let $a = 2$. Then $a^{-1} = -1/4$ satisfies the equation, but is not an integer. All three statements are false.

29. (B)

First, note that

$$P(|x - \mu| \ge c) = 1 - P(-c + \mu \le x \le c + \mu).$$

By Chebyshev's theorem, $P(\mu - k\sigma < x < \mu + k\sigma) \ge 1 - 1/k^2$. Letting $k = \sigma / c$, we find that

$$1 - P(-c + \mu \le x \le c + \mu) \le \sigma^2 / c^2.$$

Setting the right half of this inequality equal to P_0 yields $c = \sigma / \sqrt{P_0}$.

30. (A)

The probability of a head on a single toss is 1/3 and of a tail is 2/3. For two tosses, the outcomes are $\{HH, TH, HT, TT\}$. If X is the number of tails, then $P(X=0) = P(HH) = 1/9$, $P(X=1) = P(HT) + P(TH) = 4/9$, and $P(X=2) = P(TT) = 4/9$. The probability distribution is then

$$f(x) = \begin{cases} 1/9, & \text{if } x = 0; \\ 4/9, & \text{if } x = 1; \\ 4/9, & \text{if } x = 2, \end{cases}$$

$$E(X) = \sum_{i=1}^{3} x_i f(x_i) = 4/3.$$

(A)

natrix is orthogonal if $AA^T = I$. We can verify that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A) is the correct choice.

C)

g L'Hopital's rule, we find that

$$\frac{\int_{-x}^{0} \sin t^3 dt}{x^4} \to - \frac{\int_{0}^{-x} \sin t^3 dt}{x^4} \to \frac{\sin x^3}{4x^3} \to 1/4,$$

$x/x \to 1$.

C)

vector space possesses a zero element. In the case of the zero element is simply the function $f(x) = 0$. Since every C[−1, 1] that is a vector space must also contain this function, that the set in (C), which excludes the zero element, cannot or space.

34. (A)

Let $x^T = (x_1, x_2)$. For the martix in (A),

$$x^T Ax = 4x^2_1 + 4x_1x_2 + x^2_2 = (2x_1 + x_2)^2 \geq 0,$$

with equality only if x_1 and x_2 are both zero.

35. (D)

Since A and B are symmetric, we have that $(AB)^T = B^T A^T = BA$. Thus AB is symmetric if and only if $AB = BA$, i.e., that A and B commute.

36. (B)

Note that

$$\sqrt{x^2 + 2x} - x \cdot \frac{\sqrt{x^2 + 2x} + x}{\sqrt{x^2 + 2x} + x} = \frac{2}{\sqrt{1 + 2/x} + 1},$$

and the last quantity evidently tends to 1 as $x \to \infty$.

37. (E)

Using the sum-of-angles identity, note that

$$\sin[f(x)] = x^2 + \cos(\sin^{-1} x)\sin(\cos^{-1} x)$$
$$= x^2 + (\sqrt{1 - x^2})^2 = 1.$$

Thus, $f(x) = \pi/2$ for every value of x. I, II, and III are all true for constant functions.

38. (D)

The general solution to the differential equation is $y(x) = c_1 \sin|\lambda x|$ $+ c_2 \cos|\lambda| x$ where c_1 and c_2 are arbitrary constants. Now, $y(0) = 0$ implies that $c_2 = 0$. In addition, $y'(1) = 0$ implies that $\cos|\lambda| = 0$. Thus, $|\lambda| = \pi/2 + n\pi$ for every integer n, or

$$\lambda = \pm \frac{2n+1}{2}\pi,\ n = 0, 1, 2, \ldots$$

39. (E)

If $f(x)$ is everywhere differentiable, it is continuous, and since the interval is closed and bounded, $f(x)$ is uniformly continuous. Note that the existence of $f'(x)$ implies nothing about the continuity of the derivative nor the existence of the second derivative. In fact, $f'(x)$ may exist, but not be Riemann-integrable.

40. (D)

A function is periodic if $f(x+T) = f(x)$ for every x and some $T > 0$. For all the indicated functions, the innermost (first evaluated) function in the compositions is periodic, except for (D), which is not periodic. For example $\lim_{x\to-\infty} \cos[\cos(e^x)] = \cos(1)$, a constant, which cannot happen for a nonconstant periodic function.

41. (C)

First, note that

$$\left(\frac{n+2}{n+1}\right)^{2n+3} = \left(1 + \frac{1}{n+1}\right)^{2(n+1)+1}$$

Since $[1 + 1/(n+1)]^{n+1} \to e$, the right term above converges to e^2.

42. (A)

Evaluating the determinant yields the polynomial $|A| = -x^2 - x - 2$. By setting the first derivative to zero, we find that an extreme value is achieved at $x = -1/2$. This must be a maximum since the second derivative is a negative constant. Finally, note that for this value of x, $|A| = -5/4$.

43. (C)

As $t \to \infty$ the temperature approaches a steady state because there are no sources of heat and the boundary conditions are time-independent. Under this condition, the governing equation reduces to $u_{xx} = 0$, which is in fact an ordinary differential equation. The solution to the heat equation in this limit is easily seen to be $u = 10x/L$, obtained by integrating the ODE and applying the boundary conditions. The average value of u is then found from the mean value theorem for integrals to be 5.

44. (D)

First, let $P = \hat{P}_i$, where $i = 1, 2, 3$. Now, substituting this change on dependent variable into the differential equation, and linearizing gives the result that $\hat{P}_1$ and $\hat{P}_3$ become unbounded, but the $\hat{P}_2$ decays to zero. Thus, P_2 is the only stable equilibrium point.

45. (A)

By the residue theorem, $\oint_\Gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}[f(z)]$. Evaluating the residues at each singularity of $f(z)$, we find that $\mathrm{Res}[f(z)]_{z=i} = -i/2$ and $\mathrm{Res}[f(z)]_{z=-i} = i/2$. Since the sum of the residues vanishes, the integral evaluates to zero.

46. (C)

By the Cauchy Riemann equations, $u_x = v_y$ and $u_y = -v_x$. Thus, $v_y = 3y^2 - 3x^2$ so that $v(x, y) = y^3 - 3x^2y + f(x)$. From this, $v_x = -6xy + f'(x)$ which must equal $-u_y$. This last condition implies that $f(x) = -x$. Consequently, $v(x, y) = y^3 - 3x^2y - x$.

47. (A)

Evaluating the expression $|A - \lambda I| = 0$ yields the following polynomial for λ:

$$\lambda^2 - (2 + x)\lambda + 2x + 1 = 0,$$

which has the solution

$$\lambda = \frac{2 + x \pm \sqrt{(2 + x)^2 - 4(2x + 1)}}{2}$$

There will be a double eigenvalue when the discriminant is zero, that is when

$$(2 + x)^2 - 4(2x + 1) = 0.$$

The roots to this last equation are at $x = 0$ and $x = 4$.

48. (B)

Since the sequence converges, let $a = \lim_{n\to\infty} a_n$. Then $a = \sqrt{b+a}$. Because the sequence is monotone increasing, a must be positive. Solving for a, we find that

$$a = \frac{1 + \sqrt{1 + 4b}}{2}.$$

For a to be an integer, $1 + 4b$ must be an odd perfect square. Examining the odd perfect squares from 9 upward, the first value of b to occur which is a multiple of 7 is 42.

49. (A)

Using the disk method, the infintesimal volume dV of a disk centered at x is $dV = \pi[r(x)]^2dx$, where $r(x) = e^{-x}$. Thus,

$$V = \int_0^\infty dV = \int_0^\infty \pi e^{-2x}\,dx = \pi/2\,.$$

50. (E)

Counterexamples exist for all three statements. For example, the Fresnel integral $\int_0^\infty \sin(x^2)\,dx$, converges, but $\sin(x^2)$ does not coverge at all as $x \to \infty$. Thus, I may be false. By the same token, the integral just mentioned is not absolutely integrable, so II is also false. In fact, the Fresnel integral is not a square integrable either, and III is false as well.

51. (E)

Since $f(z)$ is entire, it is analytic everywhere in the complex plane. Thus $f'(z)$ is also entire. In addition, $f(z)$ has a singularity at infinity, so that no limit exists as $z \to \infty$. Also, every contour integral vanishes because $f(z)$ is everywhere analytic. $f(z) = e^z$ does not vanish at the origin.

52. (B)

Using the technique of reduction of order, a second linearly-independent solution has the form

$$y = x\left(\int^{x}\left[\frac{1}{s^2}\exp\left(-\int^{s} t\,dt\right)\right]\right).$$

This evaluates to $y = x\int^{x} e^{-t^2}/t^2\,dt$.

53. (E)

The curves are recognized as a pair of hyperbola. The general equation for this conic section is $\pm x^2/a^2 \mp y^2/b^2 = 1$. Since the hyperbolas have vertices at $(\pm 2, 0)$, it must hold that $a^2 = 4$. The only equation with the correct form is (E).

54. (C)

Note that

$$i^i = e^{(i\pi/2 + 2n\pi i)i} = e^{-\pi/2+2n\pi},$$

where n is any integer. For $n = 1$, $i^i = e^{3\pi/2}$.

(D)

onicity and continuity (or a closed and bounded interval) are conditions for integrability. In addition, all bounded and smooth functions are also integrable. However, consider the

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1/x, & \text{if } 0 < x \le 1. \end{cases}$$

is clearly not integrable, but has only one discontinuity.

56. (B)

At worst, y can be discontinuous (hence not differentiable) at every point other than zero. For example, $y = x^2$ if x is rational and $y = -x^2$ if x is irrational. However, y is differentiable at the origin. To see this, examine the difference quotient at $x = 0$:

$$-x \le (y - 0)/(x - 0) \le x.$$

As $x \to 0$, the difference quotient goes to 0 as well, showing that y' exists at the origin.

57. (D)

Take the logarithm of $f(x)$:

$$\ln f(x) = \frac{1}{\ln x} \cdot \ln x = 1,$$

so that $f(x) = e$ for $x \ne 1$. Clearly $f(1) = e$ also.

58. (D)

Applying the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|}{3}.$$

Since the limit must be less than unity for convergence, we have that $|x + 2| < 3$ or $-5 < x < 2$. At the right and left end of the interval, the series converges by the comparison test.

59. (C)

Note that this algorithm implements the formula

$$a_{n+1} = \begin{cases} a_n/2, & \text{if } a_n \text{ is even;} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$$

st value output is made *after* the input is changed, the first
es must be 40. The remaining output values are obtained by
e above formula until the output value 1 is produced at
the algorithm terminates.

ets are compact if and only if they are both closed and
) is closed but unbounded. (E) is open. (C) and (D) are
t neither open nor closed. (B) is both closed and bounded.

e have two simultaneous equations

$$\bar{6}x + \bar{5}y = \bar{0} \text{ and } \bar{8}x + by = \bar{0}.$$

x, we find that the remaining equation for y is satisfied

$$\bar{3}b = \bar{7}.$$

mod 13, $b = 11$.

y field is also an integral domain follows from the
a field. However, not every field has a finite subfield. For
rational numbers possess no finite subfield. The tran-
al numbers are not a field because 0 (the additive identity)
et.

63. (A)

Note that $27 = 2^4 + 2^3 + 2^1 + 2^0 = (011011)_2$ and $52 = 2^5 + 2^4 + 2^2 = (110100)_2$. The result digit is 1 if and only if both respective digits are 1. The result is therefore $(010000)_2 = 2^4 = 16$.

64. (B)

Every density function must integrate to unity. (E) is not integrable, while (A), (C), and (D) integrate to something other than 1. (B), in fact, is an example of an exponential distribution.

65. (C)

The contraposition of an implication is that the negation of the conclusion implies the negation of the hypothesis. Thus,

$$\neg(\neg R \wedge S) = R \vee \neg S \Rightarrow \neg(P \wedge Q) = \neg P \vee \neg Q.$$

66. (A)

First, note that the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 1$. Corresponding eigenvectors are $x^T_1 = (2, 3)$ and $x^T_2 = (1, 2)$. These eigenvalues form the columns of a matrix S which can be used to diagonalize A. The similarity matrix is therefore given by (A).