

GRE MATHEMATICS
TEST V

ANSWER KEY

1.	E	23.	E	45.	E
2.	C	24.	A	46.	A
3.	D	25.	D	47.	D
4.	B	26.	E	48.	D
5.	A	27.	E	49.	A
6.	A	28.	D	50.	B
7.	D	29.	D	51.	D
8.	E	30.	A	52.	B
9.	C	31.	D	53.	C
10.	C	32.	E	54.	D
11.	E	33.	C	55.	E
12.	D	34.	B	56.	C
13.	A	35.	A	57.	C
14.	D	36.	E	58.	A
15.	C	37.	C	59.	B
16.	A	38.	D	60.	C
17.	B	39.	E	61.	B
18.	B	40.	E	62.	D
19.	E	41.	E	63.	B
20.	C	42.	C	64.	A
21.	B	43.	A	65.	C
22.	B	44.	C	66.	B

GRE MATHEMATICS
TEST V

DETAILED EXPLANATIONS
OF ANSWERS

1. (E)

$$\text{Let } L = \lim_{n \rightarrow \infty} \left(\prod_{m=2}^n \left(1 - \frac{1}{m} \right) \right) = \lim_{n \rightarrow \infty} \left(\prod_{m=2}^n \left(\frac{m-1}{m} \right) \right)$$

Taking the natural logarithm of both sides we have

$$\ln L = \ln \lim_{n \rightarrow \infty} \left(\prod_{m=2}^n \frac{m-1}{m} \right) = \lim_{n \rightarrow \infty} \ln \left(\prod_{m=2}^n \left(\frac{m-1}{m} \right) \right)$$

We can interchange limit with \ln because $\ln x$ is continuous. Using $\ln(ab) = \ln a + \ln b$ we get

$$\begin{aligned} \ln L &= \lim_{n \rightarrow \infty} \sum_{m=2}^n [\ln(m-1) - \ln m] \\ &= \lim_{n \rightarrow \infty} (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \dots + (\ln(n-1) - \ln n) \\ &= \lim_{n \rightarrow \infty} (\ln 1 - \ln n) \\ &= \lim_{n \rightarrow \infty} (-\ln n) \\ &= -\infty, \end{aligned}$$

since $\ln 1 = 0$:

$$\Rightarrow L = \lim_{b \rightarrow \infty} e^{-b} = \lim_{b \rightarrow \infty} \frac{1}{e^b} = 0.$$

So $L = 0$.

(C)

All of the integer divisors of 2^{20} are of the form 2^k , $0 \leq k \leq 20$. Therefore, the sum of these divisors is

$$\sum_{k=0}^{20} 2^k = \frac{2^{21} - 1}{2 - 1} = 2^{21} - 1$$

Using the fact that the sum of the geometric sequence

$$\sum_{k=0}^m r^k = \frac{r^{m+1} - 1}{r - 1}.$$

(D)

The determinant of A is $\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$. Now $\det(A^{-1}) = [\det(A)]^{-1}$ using the fact that the determinant of a product is the product of their determinants.

$$\text{So } \det(A^{-1}) = (-2)^{-1} = -\frac{1}{2}.$$

(B)

The polar form of a complex number $a + ib = re^{i\theta}$ where

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Therefore $1 + i = 2^{\frac{1}{2}} e^{i\frac{\pi}{4}}$ and, taking the $4/3$ power, one obtains

$$\begin{aligned} (1 + i)^{\frac{4}{3}} &= \left(2^{\frac{1}{2}} e^{i\frac{\pi}{4}}\right)^{\frac{4}{3}} \\ &= \left(2^{\frac{1}{2}}\right)^{\frac{4}{3}} \left(e^{i\frac{\pi}{4}}\right)^{\frac{4}{3}} \\ &= 2^{\frac{2}{3}} e^{i\frac{\pi}{3}} \end{aligned}$$

$$\begin{aligned} &= 2^{\frac{2}{3}} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \\ &= 2^{\frac{2}{3}} \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right). \end{aligned}$$

5. (A)

$$\begin{aligned} 5 \log_{15} 15x - \log_{15} x^5 &= 5 [\log_{15} 15 + \log_{15} x] - 5 \log_{15} x \\ &= 5 [1 + \log_{15} x] - 5 \log_{15} x \\ &= 5 + 5 \log_{15} x - 5 \log_{15} x \\ &= 5 \end{aligned}$$

using the properties of log:

$$\log_a xy = \log_a x + \log_a y,$$

$$\log_a x^n = n \log_a x \text{ and } \log_a a = 1$$

6. (A)

By L'Hopital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{x} \int_0^x (1 + \sin t)' dt &= \lim_{x \rightarrow 0} \frac{\int_0^x (1 + \sin t)' dt}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^x (1 + \sin t)' dt}{\frac{dx}{dx}} \\ &= \lim_{x \rightarrow 0} \frac{(1 + \sin x)^x}{1} \end{aligned}$$

using the fundamental theorem of calculus:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Let
$$L = \lim_{x \rightarrow 0} (1 + \sin x)^x.$$

Taking \ln of both sides, one gets

$$\begin{aligned} \ln L &= \ln \lim_{x \rightarrow 0} (1 + \sin x)^x \\ &= \ln \lim_{x \rightarrow 0} x \ln(1 + \sin x) \\ &= \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \ln(1 + \sin x) \\ &= 0 \cdot \ln 1 \\ &= 0 \cdot 0 \\ &= 0 \end{aligned}$$

Therefore $L = e^0 = 1$.

7. (D)

Let $f(x) = \sqrt{x}$. The derivative of f at 3 is

$$f'(3) = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

But

$$f'(3) = \lim_{h \rightarrow 0} \frac{\sqrt{3+h} - \sqrt{3}}{h},$$

by the definition of derivative. Therefore,

$$\lim_{h \rightarrow 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = f'(3) = \frac{\sqrt{3}}{6}.$$

8. (E)

$\cos^2 x = \cos x$ or $\cos^2 x - \cos x = 0$ or $\cos x (\cos x - 1) = 0$ which implies $\cos x = 0$ or $\cos x - 1 = 0$. $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $\cos x = 1$ for $x = 0$ and 2π .

9. (C)

Let $B = \{x \in \mathbb{R} \mid x \text{ is an upper bound for } S\}$ where S is a set of strictly negative real numbers. B is non-empty since 0 is in B and every x in S is a lower bound of B . So B has an infimum in \mathbb{R} ; call it β . If $\gamma > \beta$, then γ is not a lower bound of B , so $\gamma \notin S$. It follows that $s \leq \beta$ for every s in S . Thus $\beta \in B$. If $\alpha < \beta$ then $\alpha \notin B$ since β is a lower bound of B . We have shown that $\beta \in B$ but $\alpha \notin B$ if $\alpha < \beta$. In other words, β is an upper bound of S , but α is not if $\alpha < \beta$. This means that $\beta = \sup S$.

10. (C)

$$\begin{aligned} \frac{\partial}{\partial y} \left(\int_0^1 e^{y \sin x} dx \right) &= \int_0^1 \frac{\partial}{\partial y} (e^{y \sin x}) dx \\ &= \int_0^1 \sin x e^{y \sin x} dx, \end{aligned}$$

using the chain rule.

11. (E)

The gradient of f , $\nabla f(x, y)$, is equal to $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$.

Differentiating $f(x, y) = \frac{2}{x^2} + 3xy$ with respect to x and y , respectively, we get

$$\frac{\partial f}{\partial x}(x, y) = -\frac{4}{x^3} + 3y \text{ and } \frac{\partial f}{\partial y}(x, y) = 3x.$$

Now

$$r^2 = \|\nabla f(r, s)\|^2 = \left(\frac{\partial f}{\partial x}(r, s)\right)^2 + \left(\frac{\partial f}{\partial y}(r, s)\right)^2,$$

by the definition of the length of a vector

$$\begin{aligned} &= \left(-\frac{4}{r^3} + 3s\right)^2 + (3r)^2 \\ &= \frac{16}{r^6} - 24\frac{s}{r^3} + 9s^2 + 9r^2 \end{aligned}$$

Multiplying both sides by r^6 we get

$$r^8 = 16 - 24r^3s + 9r^6s^2 + 9r^8.$$

That is,

$$16 - 24r^3s + 9r^6s^2 + 8r^8 = 0.$$

12. (D)

$$f(3x) = \frac{3x}{3x-1}.$$

$$\text{For (A), } \frac{3f(x)}{3f(x)-1} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-1} = \frac{3x}{2x-1}$$

$$\text{For (B), } \frac{3f(x)}{3f(x)-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3} = \frac{3x}{3} = x$$

$$\text{For (C), } 3f(x)-1 = \frac{3x}{x-1}-1 = \frac{3x-x+1}{x-1} = \frac{2x-1}{x-1}$$

$$\text{For (D), } \frac{3f(x)}{2f(x)+1} = \frac{\frac{3x}{x-1}}{\frac{2x}{x-1}+1} = \frac{3x}{2x+x-1} = \frac{3x}{3x-1}$$

$$\text{For (E), } \frac{3f(x)}{2f(x)-1} = \frac{\frac{3x}{x-1}}{\frac{2x}{x-1}-1} = \frac{3x}{2x-x+1} = \frac{3x}{x+1}$$

So, (D) is the right answer.

13. (A)

The geometric series $\sum_{k=0}^{\infty} (-x)^k$ converges to $\frac{1}{1+x}$ for $|x| < 1$.

Differentiating $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ one gets

$$-\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} k(-1)^k x^{k-1}.$$

Set $x = \frac{2}{3}$; then

$$\begin{aligned} -\frac{1}{\left(1+\frac{2}{3}\right)^2} &= \sum_{k=1}^{\infty} k(-1)^k \left(\frac{2}{3}\right)^{k-1} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n k(-1)^k \left(\frac{2}{3}\right)^{k-1} \end{aligned}$$

or

$$-\frac{9}{25} = \lim_{n \rightarrow \infty} \left[-1 + 2\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)^2 + \dots + (-1)^n n \left(\frac{2}{3}\right)^{n-1} \right] = \lim_{n \rightarrow \infty} S_n.$$

14. (D)

Since the domain of $y = f(x)$ is $0 \leq x \leq 1$, the domain of

$$f\left(x + \frac{1}{4}\right) + f\left(x - \frac{1}{4}\right) \text{ is } \begin{cases} 0 \leq x + \frac{1}{4} \leq 1 \\ 0 \leq x - \frac{1}{4} \leq 1, \end{cases}$$

$$\text{i.e., } \begin{cases} -\frac{1}{4} \leq x \leq \frac{3}{4} \\ \frac{1}{4} \leq x \leq \frac{5}{4} \end{cases}$$

$$\left[\frac{1}{4}, \frac{3}{4} \right].$$

So, the domain is

15. (C) $x + iy = z = re^{i\theta}$ where $r = \sqrt{x^2 + y^2}$

Using the polar form

$\tan^{-1}\left(\frac{y}{x}\right) = \theta$, we get $1 - i = \sqrt{2} e^{-i\frac{\pi}{4}}$. So

and

$$\begin{aligned} [\sqrt{2}(1 - i)]^{48} &= [\sqrt{2} \cdot \sqrt{2} \cdot e^{-i\frac{\pi}{4}}]^{48} \\ &= [2e^{-i\frac{\pi}{4}}]^{48} \\ &= 2^{48} e^{-i12\pi} \\ &= 2^{48} (\cos(-12\pi) + i \sin(-12\pi)) \\ &= 2^{48} \end{aligned}$$

16. (A)

$$y dx + \sqrt{x^2 + 1} dy = 0$$

$$\Rightarrow \frac{dx}{\sqrt{x^2 + 1}} + \frac{dy}{y} = 0$$

$$\Rightarrow \ln(x + \sqrt{x^2 + 1}) + \ln y = \ln C$$

$$\Rightarrow y(x + \sqrt{x^2 + 1}) = C.$$

17. (B)

The length of the arc is

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Now differentiating $x(t) = e^t \cos t$ one obtains $\frac{dx}{dt} = e^t \cos t - e^t \sin t$,

using the product rule. Similarly $\frac{dy}{dt} = -e^t \sin t - e^t \cos t$.

So

$$\begin{aligned} &\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^1 \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} dt \\ &= \int_0^1 \sqrt{e^{2t}(\cos^2 t - 2 \sin t \cos t + \sin^2 t + \cos^2 t + 2 \sin t \cos t + \sin^2 t)} dt \\ &= \int_0^1 \sqrt{2} e^t dt, \text{ after simplifying} \\ &= \sqrt{2}(e - 1). \end{aligned}$$

18. (B)

Differentiating $3x^2 + 4x^2y + xy^2 = 8$ with respect to x we get

$$6x + 8xy + 4x^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0.$$

So evaluating at $(1, 1)$ $\frac{dy}{dx} = -\frac{15}{6} = -\frac{5}{2}$. Thus the equation of the

normal line at $(1, 1)$ is $y - 1 = \frac{2}{5}(x - 1)$ (Slope of the normal line is the negative reciprocal of the slope of tangent line). To find where this line intersects the x -axis we set $y = 0$. So

$$-1 = \frac{2}{5}(x - 1) \Rightarrow -5 = 2x - 2 \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}.$$

19. (E)

$$\frac{dF}{dt}(t) = \frac{du}{dt}(t) v(t) + u(t) \frac{dv}{dt}(t),$$

by the product rule. So

$$\begin{aligned} \frac{dF}{dt}(1) &= \frac{du}{dt}(1) \cdot v(1) + u(1) \frac{dv}{dt}(1) \\ &= 2 \cdot 2 + 1 \cdot 1 \\ &= 4. \end{aligned}$$

20. (C)

The function

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x \leq 2 \\ x-6 & \text{if } 2 < x \leq 5 \\ (6-x)^2 & \text{if } 5 < x \end{cases}$$

is continuous everywhere except at the three points $x=0, 2$ and 5 . So the number of discontinuities of f is 3.

21. (B)

$$\frac{1}{x-2} < \frac{1}{x+3} \text{ or } \frac{1}{x-2} - \frac{1}{x+3} < 0, \text{ by adding } -\frac{1}{x+3}$$

$$\text{to both sides or } \frac{x+3-x-2}{(x-2)(x+3)} < 0 \text{ that is, } \frac{5}{(x-2)(x+3)} < 0$$

which implies

$$\text{either } x-2 > 0 \text{ and } x+3 < 0 \quad (1)$$

$$\text{or } x-2 < 0 \text{ and } x+3 > 0 \quad (2)$$

From relation (1) we have $x > 2$ and $x < -3$ which is impossible. From

(2) we get $x < 2$ and $x > -3$. The solution is $x \in (-3, 2)$.

22. (B)

$$\text{Given that } g\left(\frac{3+2x}{4}\right) = 1-x, \quad -\infty < x < \infty. \text{ Put } y = \frac{3+2x}{4}$$

which implies $x = \frac{4y-3}{2}$. So

$$\begin{aligned} g(y) &= 1-x \\ &= 1 - \frac{(4y-3)}{2} \\ &= \frac{2-4y+3}{2} \\ &= \frac{5-4y}{2} \end{aligned}$$

$$\text{Thus, } g\left(\frac{7z-8}{4}\right) = \frac{1}{2} \left[5 - 4\left(\frac{7z-8}{4}\right) \right], \text{ by letting } y = \frac{7z-8}{4}$$

$$= \frac{1}{2} [5 - 7z + 8]$$

$$= \frac{1}{2} [13 - 7z]$$

$$= \frac{13}{2} - \frac{7z}{2}.$$

23. (E)

$$\text{Let } e^x = u, \text{ then } f'(u) = 1 + \ln u.$$

$$\text{So } f(u) = \int (1 + \ln u) du$$

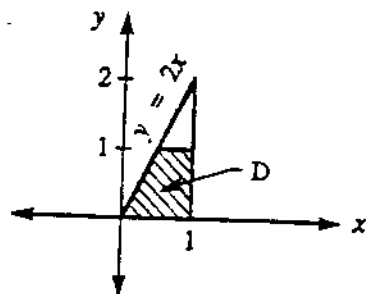
$$= u \ln u + C.$$

$$\text{Therefore, } f(x) = x \ln x + C.$$

24. (A)

Let D be the region of the integration. Then

$$D = \left\{ (x, y) : 0 \leq y \leq 1, \frac{y}{2} \leq x \leq 1 \right\}$$



So the iterated integral $\int_0^1 \int_{\frac{y}{2}}^1 e^{x^2} dx dy$ can be written as

$$\int_0^1 \int_0^{2x} e^{x^2} dy dx = \int_0^1 \int_{\frac{y}{2}}^1 e^{x^2} dx dy.$$

25. (D)

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{\left| \frac{3^{n+1}}{n+1} (x-2)^{n+1} \right|}{\left| \frac{3^n}{n} (x-2)^n \right|} \\ &= \lim_{n \rightarrow \infty} |x-2| 3 \left(\frac{n}{n+1} \right) \\ &= 3|x-2| \end{aligned}$$

By the ratio test, the series converges for

$$3|x-2| < 1 \Leftrightarrow |x-2| < \frac{1}{3}$$

$$\Leftrightarrow -\frac{1}{3} < x-2 < \frac{1}{3}$$

$$\Leftrightarrow 2 - \frac{1}{3} < x < 2 + \frac{1}{3}$$

$$\Leftrightarrow \frac{5}{3} < x < \frac{7}{3}.$$

$$\text{When } x = \frac{5}{3}, \sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{5}{3} - 2 \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the alternating series test. When

$$x = \frac{7}{3}, \sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{7}{3} - 2 \right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent harmonic series. So the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n} (x-2)^n$$

converges for x in the interval $\left[\frac{5}{3}, \frac{7}{3} \right)$.

26. (E)

The equation of the tangent plane is

$$\nabla f(-3, 2, -6) \cdot (x+3, y-2, z+6) = 0$$

where ∇f is the gradient of f and $f(x, y, z) = xz - yz^3 - yz^2 - 378$. Now differentiating f with respect to x, y and z respectively, we get

$$\frac{\partial f}{\partial x} = z$$

$$\frac{\partial f}{\partial y} = -z^3 - z^2$$

$$\frac{\partial f}{\partial z} = -3yz^2 - 2yz + x$$

Evaluating these derivatives at $(-3, 2, -6)$ one sees that $\nabla f(-3, 2, -6) = (-6, 180, -195)$. Therefore, the equation of the tangent plane is $(-6, 180, -195) \cdot (x+3, y-2, z+6) = 0$ that is, $-6x - 18 + 180y - 360 - 195z - 1170 = 0$ or $-6x + 180y - 195z - 1548 = 0$. Dividing throughout by -3 , we have $2x - 60y + 65z + 516 = 0$.

27. (E)

Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

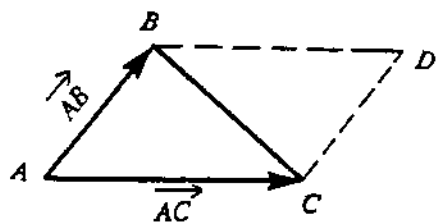
("odd terms" + "even terms"). So

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

28. (D)



The area of triangle $\triangle ABC$ is $1/2$ of the area of the parallelogram $ABDC$, which is equal to $\frac{1}{2} \|\vec{AB} \times \vec{AC}\|$, i.e., $1/2$ (the magnitude of the cross product of the vectors \vec{AB} and \vec{AC}). Now

$$\begin{aligned} \vec{AB} &= B - A = (4, 0, 2) - (2, 1, 5) \\ &= (2, -1, -3) \end{aligned}$$

and

$$\vec{AC} = C - A = (-1, 0, -1) - (2, 1, 5)$$

So

$$\begin{aligned} \vec{AB} \times \vec{AC} &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -3 \\ -3 & -1 & -6 \end{pmatrix} \\ &= 3\vec{i} - (-21)\vec{j} + (-5)\vec{k} \end{aligned}$$

Therefore,

$$\begin{aligned} \|\vec{AB} \times \vec{AC}\| &= \sqrt{3^2 + 21^2 + 5^2} \\ &= \sqrt{9 + 441 + 25} \\ &= \sqrt{475} \end{aligned}$$

Hence the area of the triangle is $\frac{\sqrt{475}}{2}$.

29. (D)

The order of a permutation σ is the least positive integer n such that $\sigma^n = \text{identity}$. Now

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

Similarly

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$$

and

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Therefore the order of σ is 4.

30. (A)

If \vec{u} and \vec{v} are orthogonal vectors, then their dot product is zero, that is $\vec{u} \cdot \vec{v} = 0$ and therefore $r\vec{u} \cdot s\vec{v} = (rs)(\vec{u} \cdot \vec{v}) = 0$ for all real r and s .

31. (D)

$x \in A \cap B$ then $x \in A$ and $x \in B$.

32. (E)

First observe that R is an abelian group under the $*$ operation with identity element 0 and inverse of every $a \neq -1$ in R given by $-\frac{a}{1+a}$. Therefore

$$\begin{aligned} 7 &= 5 * x * 3 \\ &= x * 5 * 3 \quad \text{by using the abelian property} \\ &= x * (5 + 3 + 15) \quad \text{by using the definition of } * \\ &= x * (23). \end{aligned}$$

Thus

$$\begin{aligned} 7 * \left(-\frac{23}{24}\right) &= x * 23 * \left(-\frac{23}{24}\right) \\ &= x * 0 \quad \text{as } 23 * \left(-\frac{23}{24}\right) = 23 - \frac{23}{24} - \frac{23 \cdot 23}{24} = 0 \\ &= x. \end{aligned}$$

Hence
$$x = 7 * \left(-\frac{23}{24}\right) = 7 - \frac{23}{24} - 7 \cdot \frac{23}{24} = \frac{168 - 23 - 161}{24}$$

by simplifying
$$= -\frac{16}{24} = -\frac{2}{3}.$$

33. (C)

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^k + \dots \quad \text{for } |x| < 1.$$

Differentiating both sides we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + kx^{k-1} + (k+1)x^k + \dots \quad \text{for } |x| < 1.$$

Differentiating two more times we obtain

$$\frac{2}{(1-x)^3} = 2 + 3 \cdot 2x + \dots + (k+1)kx^{k-1} + (k+2)(k+1)x^k + \dots$$

and

$$\frac{6}{(1-x)^4} = 3 \cdot 2 + 4 \cdot 3 \cdot 2x + \dots + (k+3)(k+2)(k+1)x^k + \dots$$

or

$$\begin{aligned} \frac{1}{(1-x)^4} &= \frac{3 \cdot 2}{6} + \frac{4 \cdot 3 \cdot 2}{6}x + \dots \\ &+ \frac{(k+3)(k+2)(k+1)}{6}x^k + \dots \end{aligned}$$

by dividing by 6. Now

$$\begin{aligned} (1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots)^2 &= \left[\frac{1}{(1-x)^2} \right]^2 \\ &= \frac{1}{(1-x)^4} \\ &= \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{6} x^k \end{aligned}$$

Therefore

$$b_k = \frac{(k+3)(k+2)(k+1)}{6}.$$

3)

continuous since $g'(x)$ exists for all real x . By the mean value theorem there exist x_1 and x_2 , $a < x_1 < b < x_2 < c$ such that

$$g'(x_1) = \frac{g(b) - g(a)}{b - a}$$

$$g'(x_2) = \frac{g(c) - g(b)}{c - b}$$

Since $g(b) = g(c) = 0$, therefore $g'(x_1) = 0 = g'(x_2)$ and hence the only possible zeros for $g'(x)$ is 2.

A)

Since that

$$\text{if } 0 \leq x \leq \frac{\pi}{4}, \text{ then } \sin x - \cos x \leq 0,$$

$$\text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2}, \text{ then } \sin x - \cos x \geq 0,$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} |\sin x - \cos x| dx &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \\ &\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx \\ &= 2\sqrt{2} - 2. \end{aligned}$$

E)

The set of integers under subtraction is not a group because $(a - b) - c \neq a - (b - c)$, that is associativity fails.
The set of non-zero real numbers under division is not a group.
The set of even integers under addition is a group.
The set of all integer multiples of 13 is group under addition.

37. (C)

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix} = -6\vec{i} + 8\vec{k} - 5\vec{j} - 3\vec{k} - 20\vec{i} - 4\vec{j} \\ &= -26\vec{i} - 9\vec{j} + 5\vec{k}. \end{aligned}$$

38. (D)

Matrix multiplication is defined whenever the number of columns of the first is equal to the number of rows of the second. Thus only TS and ST are defined.

39. (E)

The system of equations

$$\begin{aligned} px + y &= 1 \\ x + py &= 2 \\ y + pz &= 3 \end{aligned}$$

has no solutions if and only if the determinant of the coefficient matrix is zero. Thus

$$\begin{aligned} \det \begin{pmatrix} p & 1 & 0 \\ 1 & p & 0 \\ 0 & 1 & p \end{pmatrix} &= p(p^2 - 0) - 1(p - 0) + 0(1 - 0) \\ &= p(p^2 - 1) \\ &= p(p + 1)(p - 1) \end{aligned}$$

which is zero when $p = 0, 1$ or -1 .

(E)

The determinant of the given matrix is always zero for any value of x , because the determinant of a lower triangular matrix is the product of its diagonal elements, which in this case is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 0 = 0$.

(E)

By partial fractions decomposition

$$f(x) = \frac{1}{1-x^2} = \frac{1}{2} \left[\frac{1}{1-x} + \frac{1}{1+x} \right].$$

The n th derivative of $\frac{1}{1-x}$ is $\frac{n!}{(1-x)^{n+1}}$ and the n th derivative

of $\frac{1}{1+x}$ is $\frac{(-1)^n n!}{(1+x)^{n+1}}$. Therefore

$$f^{(n)}(x) = \frac{n!}{2} \left[\frac{1}{(1-x)^{n+1}} + \frac{(-1)^n}{(1+x)^{n+1}} \right].$$

(C)

The number of different partial derivatives of order k is equal to the number of distinct nonnegative integer-valued vectors $(\alpha_1, \alpha_2, \dots, \alpha_n)$ satisfying $\alpha_1 + \alpha_2 + \dots + \alpha_n = k$. There are

$$\binom{k+n-1}{k}$$

such vectors. Thus the number of different partial derivatives of order

$$\binom{k+n-1}{k}$$

43. (A)

$$\begin{aligned} T(4e^{2x} + 7e^{3x} - 5) &= 4T(e^{2x}) + 7T(e^{3x}) - 5T(1) \text{ by linearity} \\ &= 4 \sin x + 7 \cos 4x - 5e^{5x}. \end{aligned}$$

44. (C)

First school has $\binom{8}{2}$ choices, second school has $\binom{6}{2}$ choices, third school has $\binom{4}{2}$ choices and last school has to accept the remaining 2. Therefore 8 teachers can be divided into 4 schools in

$$\binom{8}{2} \binom{6}{2} \binom{4}{2} \binom{2}{2} = 2520$$

ways if each school must receive 2 teachers.

45. (E)

The Cauchy integral formula states that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$$

where z_0 is an interior point of C . In the case at hand $f(z) = 2z^2 - z - 2$ which is analytic and $z_0 = 2$ which lies inside the circle $|z| = 3$. Observe that $f(2) = 4$. Therefore

$$g(2) = \int_C \frac{f(z)}{z-2} dz = 2\pi i f(2) = 8\pi i.$$

46. (A)

Factoring out -2 in $-2 + 1 - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} \dots$ one gets

$$\begin{aligned}
 & -2 + 1 - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \dots \\
 & = -2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right) \\
 & = -2 \ln 2.
 \end{aligned}$$

dimension of the eigenspace corresponding to the eigenvalue number of linearly independent solutions of the homogeneous equations $(A - \lambda I)\bar{X} = 0$. This is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 \\ -4 & -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using matrix multiplication this reduces to

$$\begin{aligned}
 0 &= 0 \\
 -2x &= 0 \\
 -3x - 2y &= 0 \\
 -4x - 3y - 2z + w &= 0.
 \end{aligned}$$

$x=0$ implies $x=0$ and $-3x-2y=0$ and $x=0$ implies $y=0$. Using $x=0$ and $y=0$, in the fourth equation we have $w=2z$. Thus only one solution

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

the dimension of the eigenspace is 1.

48. (D)

Since T is onto, the dimension of the range space of T is equal to the dimension of W , which is 7. By the rank-nullity theorem we have $\dim V = \dim$ of the null space + \dim of range space. Therefore the dimension of the null space of T is equal to $11 - 7 = 4$.

49. (A)

$$\begin{aligned}
 \vec{u} \cdot \vec{v} &= (x, y) \cdot (x', y') \\
 &= xx' + yy'.
 \end{aligned}$$

50. (B)

Differentiating

$$F(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

with respect to x we get the density function

$$f(x) = F'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

for $x > 0$ and 0 for $x < 0$. The mean is

$$\begin{aligned}
 E(\bar{X}) &= \int_{-\infty}^{\infty} x f(x) dx \\
 &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} 2x e^{-x^2} dx \\
 \int_0^{\infty} 2x e^{-x^2} dx &= \lim_{b \rightarrow \infty} \int_0^b 2x e^{-x^2} dx \\
 &= \lim_{b \rightarrow \infty} 1 - e^{-b^2} = 1
 \end{aligned}$$

Therefore the mean is $\frac{1}{\sqrt{\pi}}$.

51. (D)

The characteristic polynomial of A is given by

$$\begin{aligned} P_A(x) &= \det(A - xI) = \det \begin{bmatrix} 1-x & 1 \\ 1 & 2-x \end{bmatrix} \\ &= (1-x)(2-x) - 1 \\ &= 2 - 3x + x^2 - 1 \\ &= x^2 - 3x + 1. \end{aligned}$$

By the Cayley-Hamilton theorem $P_A(A) = 0$; that is $A^2 - 3A + I = 0$.

52. (B)

When $t = 2$, $x = 2^3 - 4 = 4$, $y = 2 \cdot 2^2 + 1 = 9$. The slope of the tangent line is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4}{3t}.$$

So $\frac{dy}{dx} \Big|_{t=2} = \frac{2}{3}$, this equation is $y - 9 = \frac{2}{3}(x - 4)$
or $2x - 3y + 19 = 0$.

53. (C)

$$\begin{aligned} \int_0^1 x f''(2x) dx &= \frac{1}{2} \int_0^1 x d[f'(2x)] \\ &= \frac{1}{2} \left[x f'(2x) \Big|_0^1 - \int_0^1 f'(2x) dx \right] \\ &= \frac{1}{2} \left[5 - \frac{1}{2} f(2x) \Big|_0^1 \right] \\ &= \frac{1}{2} \left[5 - \frac{1}{2} f(2) + \frac{1}{2} f(0) \right] \\ &= 2 \end{aligned}$$

54. (D)

Let four consecutive integers be $n, n + 1, n + 2$, and $n + 3$, then

$$\begin{aligned} n(n+1)(n+2)(n+3) + 1 &= n^4 + 6n^3 + 11n^2 + 6n + 1 \\ &= n^4 + 2n^2(3n+1) + (9n^2 + 6n + 1) \\ &= n^4 + 2n^2(3n+1) + (3n+1)^2 \\ &= (n^2 + 3n + 1)^2. \end{aligned}$$

55. (E)

The auxiliary equation is $r^2 + 5r + 6 = 0$ whose solutions are $r = -2$ and -3 . Thus the solution of the differential equation is $y(x) = c_1 e^{-2x} + c_2 e^{-3x}$. Using the initial conditions we get $c_1 + c_2 = 0$ and $-2c_1 - 3c_2 = 1$, whose solution is $c_1 = 1$ and $c_2 = -1$. Hence the solution of the differential equation satisfying the initial conditions is given by $y(x) = e^{-2x} - e^{-3x}$.

56. (C)

$z + \sin z$ and ze^z are analytic functions.

57. (C)

We know that if p is a prime number, then $n^p \equiv n \pmod{p}$ for any positive integer n . i.e., $n^p - n \equiv 0 \pmod{p}$. In this problem, 7 is prime, so the answer is (C).

58. (A)

$$f = (1478) (265) (39) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 9 & 7 & 2 & 5 & 8 & 1 & 3 \end{pmatrix}$$

$$\begin{aligned} \therefore f^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 9 & 1 & 6 & 2 & 4 & 7 & 3 \end{pmatrix} \\ &= (1874) (256) (39). \end{aligned}$$

59. (B)

If f is a probability density function then

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

So

$$c \int_0^{\infty} x e^{-x^2} dx = 1. \text{ Now } \int_0^{\infty} x e^{-x^2} dx = \frac{1}{2} \int_0^{\infty} 2x e^{-x^2} dx.$$

Let $u = x^2$ then $du = 2x dx$ and

$$\int_0^{\infty} x e^{-x^2} dx = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}.$$

Therefore $\frac{c}{2} = 1$ which implies $c = 2$.

60. (C)

$$\begin{aligned} &\neg [\forall x \exists y (P(x, y) \wedge \neg Q(x, y))] \\ &= \exists x \forall y (\neg P(x, y) \vee Q(x, y)) \\ &= \exists x \forall y (P(x, y) \rightarrow Q(x, y)). \end{aligned}$$

61. (B)

The Lebesgue measure of a countable set is always zero.

62. (D)

The ratio test gives

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{(3-x)(6x-7)^{n+1}}{(3-x)(6x-7)^n} \right| \\ &= |6x-7|. \end{aligned}$$

So the series converges for $|6x-7| < 1$, that is $-1 < 6x-7 < 1$. Then dividing by 6 we have $1 < x < \frac{4}{3}$. Now we check the end points.

When $x = 1$ the series

$$\sum_{n=0}^{\infty} (3-1)(6-7)^n = \sum_{n=0}^{\infty} 2(-1)^n$$

diverges by the n th term test. When $x = \frac{4}{3}$, the series

$$\sum_{n=0}^{\infty} \left(3 - \frac{4}{3}\right) \left(6 \cdot \frac{4}{3} - 7\right)^n = \sum_{n=0}^{\infty} \frac{5}{3} (1)^n$$

which diverges again by the n th term test. Thus, the interval of convergence is $(1, \frac{4}{3})$.

63. (B)

Since the coefficient matrix A has non-zero determinant, the system has a unique solution.

64. (A)

$$f(x) = f_1(x) + f_2(x) \quad \text{where} \quad f_1(x) = \frac{1}{2} [f(x) + f(-x)]$$

$$\text{and} \quad f_2(x) = \frac{1}{2} [f(x) - f(-x)].$$

Thus for the given function $f(x) = x^4 + x^6 + \sin(x^2) + e^{x^3}$. We have

$$\begin{aligned} f_2(x) &= \frac{1}{2} [x^4 + x^6 + \sin(x^2) + e^{x^3} - x^4 - x^6 - \sin((-x)^2) - e^{-x^3}] \\ &= \frac{1}{2} (e^{x^3} - e^{-x^3}). \end{aligned}$$

65. (C)

The sum of the geometric sequence $1 + w + w^2 + \dots + w^{n-1} = \frac{1 - w^n}{1 - w}$ and hence using the fact that w is n th root of unity, $w^n = 1$, we have $1 + w + w^2 + \dots + w^{n-1} = 0$. Therefore $w + w^2 + \dots + w^{n-1} = -1$.

66. (B)

$$\begin{aligned} P(A \cup B) &= 1 - P(\overline{[A \cup B]}) \\ &= 1 - 0.1 \\ &= 0.9 \end{aligned}$$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and hence

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.7 + 0.5 - 0.9 \\ &= 0.3. \end{aligned}$$

Therefore $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = \frac{3}{5}$.

GRE

MATH TEST

TEST VI