GRE MATHEMATICS
TEST IV

ANSWER KEY

1.	A	23.	A	45.	D
2.	E	24.	B	46.	E
3.	B	25.	E	47.	C
4.	A	26.	B	48.	D
5.	D	27.	A	49.	D
6.	C	28.	C	50.	C
7.	D	29.	B	51.	E
8.	D	30.	E	52.	C
9	E	31.	C	53.	E
10.	E	32.	B	54.	D
11.	C	33.	A	55.	D
12.	D	34.	A	56.	E
13.	E	35.	C	57.	A
14.	B	36.	B	58.	D
15.	C	37.	B	59	B
16.	A	38.	C	60.	A
17.	A	39.	D	61.	B
18.	A	40.	A	62.	D
19.	E	41.	E	63.	D
20.	D	42.	D	64.	C
21.	A	43.	D	65.	A
22.	B	44.	A	66.	B

GRE MATHEMATICS
TEST IV

DETAILED EXPLANATIONS
OF ANSWERS

1.　(A)

When $\bar{x}$ is an eigenvector for matrix, A, then $A\bar{x} = \lambda\bar{x}$ defin the eigenvalue, λ. In this case,

$$A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ so } \lambda = 5.$$

2.　(E)

The other vertex is $(x, 0, 3)$ and so, using dot products,

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{3}{4}}$$

$$= \frac{x^2 + 9}{\sqrt{x^2 + 13}\sqrt{x^2 + 9}}$$

$$= \sqrt{\frac{x^2 + 9}{x^2 + 13}}$$

Then $4(x^2 + 9) = 3(x^2 + 13)$ implies $x^2 = 3$ and $x = \sqrt{3}$.
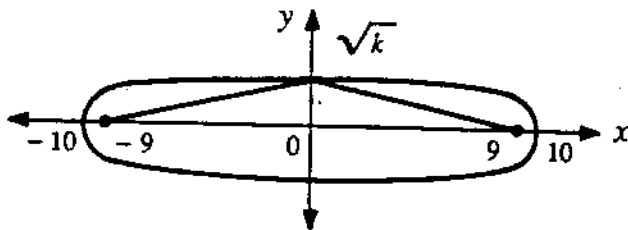
3.　(B)

"Uniform" means that each value has equal probabilty, $\frac{1}{5}$. Then variance of x is

$$E(x^2) - (E(x))^2 = \sum_{x=1}^{5} \frac{x^2}{5} - \left(\sum_{x=1}^{5} \frac{x}{5}\right)^2$$

$$= 11 - 3^2$$

$$= 2.$$

4.　(A)

The parabola is $y^2 = x - 1$ (P) and the hyperbola is $2xy = 1(H)$. Since the shaded region is above (H) we must have $y > 1/2x$ or $2x > 1/y$. And, since the shaded region is to the right of (P) we must have $x > y^2 + 1$ or $y^2 + 1 < x$. As usual "and" represents intersection.

5.　(D)

The vertices from standard form are as shown. The statement implies that the foci must be at $(\pm 9, 0)$. The locus definition of an ellipse implies that $20 = 2\sqrt{k + 9^2}$ so that $100 = k + 9^2$ and $k = 19$.

6.　(C)

Fermat's Little Theorem states that $a^{p-1} \equiv 1 \pmod{p}$ for any prime p and any a which is not divisible by p. Here, the form indicates $p = 11$.

7.　(D)

Since permutations are functions, the "product" is actually a composition and proceeds from right to left. So, $1 \to 3 \to 4$, $2 \to 5 \to 3$, $3 \to 2 \to 1$, $4 \to 1 \to 2$, and $5 \to 4 \to 5$. We get (4 3 1 2 5).

8.　(D)

One can apply DeMorgan's Law which states $\overline{A \cap B} = \overline{A} \cup \overline{B}$. In this case $A = \overline{S}$ so $\overline{A} = \overline{\overline{S}} = S$.

9.　(E)

The ratio test, $\rho = \lim_{n \to \infty} \frac{|x - 3|^2 n}{4 \cdot (n + 1)} = \left|\frac{x - 3}{2}\right|^2 < 1$, implies $|x - 3| < 2$ or $-2 < x - 3 < 2$ and so $1 < x < 5$. Finally, checking endpoints directly we find for $x = 1$ that

$$2\sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n}$$

converges by the alternating series (Leibniz) test and for $x = 5$ that

$$2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

likewise converges.

10.   (E)

There are two possible parabolic arcs that will contain $(4, 2)$. Either $f(x) = \sqrt{x}$ or $g(x) = \dfrac{x^2}{8}$ will do. $(16, 2)$ lies on the graph of $f(f(x)) = x^{1/4}$ and $(8, 8)$ lies on $g(g(x)) = \dfrac{x^4}{2^9}$.

11.   (C)

The effective rate is $(1 + .01)^4 - 1 = .04060401 \doteq 4.06\%$.

12.   (D)

The symmetric limit for the derivative at $x = 2$ has the form
$$\lim_{x \to 0} \frac{f(2 + x) - f(2 - x)}{2x} = f'(2) = \frac{1}{4}$$
since the tangent line slope is the negative reciprocal of the normal line slope. Since our limit has a reversed numerator and only $\dfrac{1}{2}$ the denominator, it has value $-\dfrac{1}{2}$.

13.   (E)

A single eigenvalue must then be of multiplicity 3 for a 3 x 3 matrix. In general the dimension of the eigenspace may be any number greater than zero and less than or equal to the multiplicity.

14.   (B)

Since $f(e^x) = \sqrt{x}$ we must have $f(x) = f(e^{\log x}) = \sqrt{\log x}$ and here is where $x \geq 1$ is needed for definition of the radical. Since $y = \sqrt{\log x}$ may be solved for $x = e^{y^2} = f^{-1}(y)$ we find that $f^{-1}(x) = e^{x^2}$.

15.   (C)

The $*$–identity is 0 since $a*0 = 0*a = a$ for each rational, a. Then the $*$–inverse of a is found by solving $a*x = a + x - ax = 0$ for $x = \dfrac{a}{(a - 1)}, a \neq 1$. Now x is an integer when $a = 0$ and $a = 2$ and in no other case.

16.   (A)

In general
$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x \quad \text{where} \quad \Delta x = \frac{(b - a)}{n}.$$
This may be called the limit of the right end point Riemann sum. In this case we let $f(x) = x^2$, $a = 1$, and $b = 3$ so $\Delta x = \dfrac{2}{n}$. It follows by the fundamental theorem of calculus that the limit is
$$\int_1^3 x^2 dx = \left.\frac{x^3}{3}\right|_1^3 = 9 - \frac{1}{3} = \frac{26}{3}.$$

17.   (A)

T is representable as a 2 x 3 matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

Then S is an inverse if $TS = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The given transformation for S are representable as 3 x 2 matrices

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \\ \frac{1}{3} & 0 \end{bmatrix} \text{ respectively.}$$

Direct multiplications show that only (A) is inverse.

(A)

The test for a homomorphism is to show that

$$\left(gh^{-1}\right)\phi = (g\phi)(h\phi)^{-1} \quad \text{for arbitrary } g, h \in G.$$

this case

$$_{)}(h\phi)^{-1} = a^2 g a^2 (a^2 h a^2)^{-1} = a^2 g a^2 a^{-2} h^{-1} a^{-2} = a^2\left(gh^{-1}\right)a^{-2}$$

ich will be $(gh^{-1})\phi$ as desired if $a^2 = a^{-2}$ or $a^4 = e$.

(E)

The entry is the cofactor of the entry in row 2 and column 3 divided the determinant of A. The cofactor is

$$(-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \quad \text{and} \quad |A| = -4.$$

(D)

It is well-known that $f(A \cup B) = f(A) \cup f(B)$. (A) is contradicted a constant function with disjoint A and B. Well-chosen constant nctions and sets will also contradict (B) and (C).

(A)

$$\cos^2 nx = \frac{1 + \cos^2 nx}{2} \quad \text{with integral} \quad \left(\frac{x}{2} + \frac{\sin^2 nx}{4n}\right)\Big|_0^\pi = \frac{\pi}{2}$$

nce $\sin^2 n\pi = \sin 0 = 0$ for all n.

(B)

If we represent the polynomial by $p(x)$ then in (A) we find $p(1)$ $4 = 0 \pmod 4$ so $p(x)$ is reducible. In (B) we find $p(0) = 3$, $p(1)$ 1, $p(2) = 1$, and $p(3) = 3$ modulo four and so $p(x)$ is irreducible.

23.   (A)

The circumference is $2\pi\left(\frac{4}{\pi}\right) = 8$ and so the chord of an arc of length 2 is the hypotenuse of a right triangle of side lengths $\frac{4}{\pi}$ and area $\frac{1}{2}\left(\frac{4}{\pi}\right)^2 = \frac{8}{\pi^2}$. The area of the quarter-circle is $\frac{\pi}{4}\left(\frac{4}{\pi}\right)^2 = \frac{4}{\pi}$ and the desired area is the difference

$$\frac{4}{\pi} - \frac{8}{\pi^2} = \frac{4}{\pi}\left[1 - \frac{2}{\pi}\right].$$

24.   (B)

The limit function $f(x) = 0$ for all x in $[0, 1)$ and $f(1) = 1$. Since the uniform limit of continuous functions is always continuous, (C) is impossible.

25.   (E)

Dividing both numerator and denominator by e^x results in a determinate form.

26.   (B)

Since the third vector is a linear combination of the two elementary basis vectors, the column rank is two. The dimension of the solution space must be $6 - 2 = 4$.

27.   (A)

The rate of change is the directional derivative,

$$(f_x\hat{i} + f_y\hat{j}) \cdot \frac{(3\hat{i} - 4\hat{j})}{5} = \frac{(6x - 8y)}{5} = -\frac{10}{5} = -2$$

at $(1, 2, 5)$.

28. (C)

A_4 is the subgroup of even permutations in S_4. It is easy to inspect that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = e \text{, so (i) is odd.}$$

Likewise $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}^2 = e$, so (ii) is even

and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}^3 = e$, so (iii) is odd. The product of odd and even is odd and the product of odd and odd is even so on (i), (iii) will product an even product.

29. (B)

By partially integrating the given derivatives we find at first that

$$f(x,y) = \int (3x^2y^2 + y^4)dx = x^3y^2 + xy^4 + g(y)$$

where $g(y)$ can be any function of y. Finding

$$\frac{\partial f}{\partial y} = 2x^3y + 4xy^3 + g'(y)$$

from this representation implies that

$$g'(y) = 0 \text{ so } g(y) = C.$$

So $f(x,y) = x^3y^2 + xy^4 + C = xy^2(x^2 + y^2) + C$.

30. (E)

The distance (directed) from the yz plane to any point (x,y,z) is the coordinate $x = r\cos\theta$ in cylindrical coordinates. The representation of dV is $r\,dz\,dr\,d\theta$. The integrand must be $r^2\cos\theta$.

31. (C)

The equation is separable as $x\,dx = \dfrac{dy}{(y^2+1)}$ and solved as

$\dfrac{x^2}{2} = \tan^{-1}y + C$ where $y(0) = -1$ implies $C = \dfrac{\pi}{4}$. Then $x = \sqrt{\pi}$ implies $\tan^{-1}y = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$ and $y = 1$.

32. (B)

The point $(0, 0)$ connects the well known graph of $x\sin\left(\dfrac{1}{x}\right)$ as shown. Since the domain is $(-\infty, \infty)$, the graph cannot be compact. It is a closed subset since the only missing limit point on the graph of $x\sin\left(\dfrac{1}{x}\right)$ is $(0, 0)$.
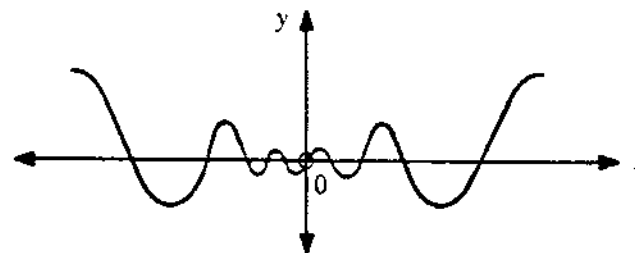
33. (A)

As subtasks we may first choose any one of 13 denominations and then find 4 different subsets of three of the same denomination. Finally the number of ways to choose the remaining two cards from the 48 not in the selected denomination is $48 \cdot 47/2$. Division by two is required as subsets are desired and so orders of selection should not be counted. The counting principle for products implies the solution $13 \cdot 4 \cdot 14 \cdot 47 = 24 \cdot 47 \cdot 52$.

34. (A)

In standard form, $y' - \dfrac{2}{x}y = x$ and so the integrating factor is

$$e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

35. (C)

Since $(3, -2, 1) = 3(1, 0, 0) - 2(0, 1, 0) + 1(0, 0, 1)$ its image must be $3(1, 2, 3) - 2(2, 3, 1) + (1, 1, -2) = (0, 1, 5)$.

36. (B)

The 9th partial sum of a convergent alternating series with decreasing absolute terms has absolute error less than the 10th term which is $1/100$. Since the approximation ends with a negative term it is less than the sum of the series and so $0 < E < .01$.

37. (B)

In powers of 2 we find $73 = 64 + 8 + 1 = 2^6 + 2^3 + 2^0$, so the binary representaiton is $1\ 0\ 0\ 1\ 0\ 0\ 1$.

38. (C)

$f'(x) = 3x^2 - 6x = 3x(x - 2)$ and so $x = 2$ is the only critical point in $[1, 3]$. Now $f(1) = k - 2$, $f(2) = k - 4$, and $f(3) = k$. We must require $k = k - 4$ or $k = 4 - k$ and only the latter can be solved with $k = 2$.

39. (D)

In general an n-gon must have $\binom{n}{2} - n = \dfrac{n(n-3)}{2}$ diagonals. Note that $\binom{n}{2}$ counts the number of pairs of vertices and all but n of these pairs (sides) represent diagonals. Now to have 10 times as many diagonals requires $\dfrac{(n-3)}{2} = 10$ and $n = 23$.

40. (A)

All three triangles are similar since they are right triangles sharing an acute angle. The dimensions of the small triangles are in ratio $3:4$ as seen in the hypotenuses. Hence the areas must be in ratio $\left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$.

41. (E)

Not considering the circular layout there are $6! = 720$ permutations. These can be grouped into 6 cyclic permutations per group each representing equivalent circular arrangements. So we get $\dfrac{6!}{6} = 120$.
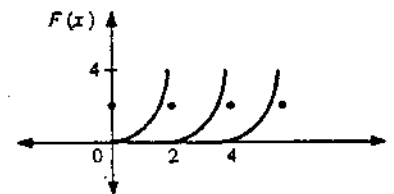
42. (D)

The average value is defined as the area under the curve (in this case a quarter circle of radius 2) divided by the base interval length. We get $\dfrac{\pi 2^2}{8} = \dfrac{\pi}{2}$. The integration approach to the area is much harder than this approach.
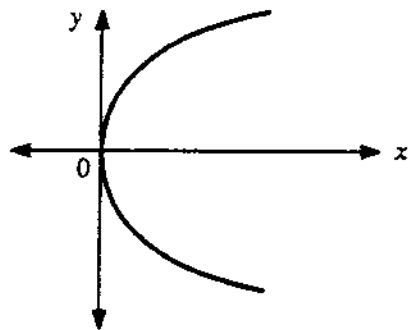
43. (D)

It is well known that $F(x)$ converges to the average of its left and right hand limits at all points. Here,

$$F(4) = \frac{F(4^-) + F(4^+)}{2} = \frac{0 + 4}{2} = 2.$$

f course many functions satisfy the coordinate criteria but all
have limit zero at $x = 0$. An example where this is the only limit
$x) = \sqrt{x}$ if x is rational but $f(x) = -\sqrt{x}$ if x is irrational.
ints lie on the parabola $x = y^2$.

---

unded sequences always contain convergent subsequences and
ated sequence in (C) is always convergent. Any convergent
uence contains infinitely many subsequences all converging to
ne limit. Since $\{x^n\}$ is divergent we must have different limits
n x_k and $\max_{k \le n} x_k$.

---

implies $\overline{s}$ " is the negation of "r implies s" and is equiva-
its converse which is not generally true. The contrapositive of
plies $\overline{q}$ and $\overline{r}$" is "q or r implies p" and these are equiva-
o q implies p as q satisfies "q or r" and hence I. is true. Also
es p and, via the transitivity property, since p implies q we must
r implies q" and III. is true.

---

47.   (C)

I.   follows as true since in general $|ab| = |a||b|$.

II. is also true since $|x| \ne |y|$ implies that $|x + y| > 0$.

III. is not generally true. The triangle inequality states that

$|x + y| \le |x| + |y|$ in general but $|x - y| \ge \dfrac{1}{(|x| + |y|)}$

fails to imply that $|x - y| \ge \dfrac{1}{|x + y|} \ge \dfrac{1}{(|x| + |y|)}$.

For example, $x = \dfrac{4}{5}$ and $y = -\dfrac{3}{5}$ finds $|x - y| < \dfrac{1}{|x + y|}$.

Similarly, $|x - y| \ge \dfrac{1}{4}$ fails to imply $|x - y| \ge \dfrac{1}{(|x| + |y|)}$

so IV. is false.

---

48.   (E)

Let N represent the event "not one, two, or six" and let S represent
the event "six." Then the probability is the sum of the probabilities of
the sequences of events, $NN...NS$, where the number of N's varies as
$0,1,2,...,k,....$ The sequence probabilities are $\left(\dfrac{1}{2}\right)^k\left(\dfrac{1}{6}\right)$ and the sum
is geometric with value $\left(\dfrac{1}{6}\right)\left(1 - \dfrac{1}{2}\right) = \dfrac{1}{3}$.

---

49.   (D)

Using $F(x,y,z) = x^2 z - 2yz^2 + xy$ we find

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{[x^2 - 4yz]}{[2xz + y]}.$$

At $(1,1,1)$ the value is $-\dfrac{(-3)}{3} = 1$.

50. (C)

The general idea is that inverses of products are products of inverses in the reverse order.

51. (E)

A left ideal is a subring with the additional property that for each $r \in R$ and $h \in H$ we have $rh \in H$ so $RH \subseteq H$. Since $eH = H$ we have $H \subseteq RH$ so $RH = H$. For any subring, $HH = H$.

52. (C)

This is a fundamental result of real analysis. The example with $C_n = \left[0, \frac{1}{n}\right]$ yields singleton $S = \{0\}$ which is a counterexample to (D). Boundedness is necessary or $C_n = [n, \infty)$ implies $S = 0$.

53. (E)

Only 10 values of $X(I)$ are examined since $J = 11$ initiates a return. $M = 0$ is replaced only when $X(J) > 0$ so negative entries in the array are not further processed. Whenever a greater positive value of $X(J)$ occurs, M takes this value. Hence the last statement is strongest as minimums or maximums in general may be negative and a strictly positive array is not required as in (C).

54. (D)

Simplest is a conditional probability approach on the second ball drawn when two colors are permitted (20 balls) out of the remaining 29 balls following the first draw.

55. (D)

By inspection, the sum of the coefficients is zero so 1 is a root. Synthetic division implies the quadratic quotient is $x^2 - 2x + 3$ with roots

$$\frac{(2 \pm \sqrt{-8})}{2} = 1 \pm \sqrt{2}\,i .$$

56. (E)

The main theoretical point is that the coefficient of $(x - 2)^n$ is $\dfrac{f^{(n)}(2)}{n!}$ and here, with $n = 5$, we get

$$\frac{1}{32 \cdot 6} = \frac{f^{(5)}(2)}{5!} \quad \text{so} \quad f^{(5)}(2) = \frac{120}{32 \cdot 6} = \frac{5}{8} .$$

57. (A)

Since our function is "even" we know that $f(-7) = f(7)$.

58. (D)

$$\left| 2(AB)^{-1} \right| = \frac{2^4}{|AB|} = \frac{2^4}{(|A||B|)} = \frac{2^4}{6} = \frac{8}{3} .$$

59. (B)

Using modular arithmetic, we find that

$$7x + 3y \equiv 0 \ (\text{mod } 13) \quad \text{and} \quad 8x + ky \equiv 0 \ (\text{mod } 13),$$

so by subtraction

$$x + (k - 3)y \equiv 0 \ (\text{mod } 13) .$$

Multiplying by 6, we have

$$6x + 6(k - 3)y \equiv 0 \ (\text{mod } 13) .$$

ling $7x + 3y \equiv 0 \pmod{13}$ gives

$$13x + 6(k-3)y + 3y \equiv 0 \pmod{13}$$

$$\Rightarrow 13x + [6(k-3) + 3]y \equiv 0 \pmod{13}.$$

$$\Rightarrow (6k - 15)y \equiv 0 \pmod{13}$$

$$\Rightarrow (6k - 2)y \equiv 0 \pmod{13}$$

e y can vary, we require

$$(6k - 2) \equiv 0 \pmod{13}$$

by inspection of the choices, we find that $k = 9$ works, since

$$6 \cdot 9 - 2 = 54 - 2 = 52 = 13 \cdot 4 \equiv 0 \pmod{13}.$$

(A)

The approximating form is $y_{n+1} = y_n + y'(x_n)\Delta x$.
is case

$$\Delta x = 0.1, x_0 = y_0 = 1 \text{ and } y' = \frac{(x+y)}{x}.$$

$$y_1 = 1 + (2)(0.1)$$

$$= 1.2.$$

$$y_2 = 1.2 + \left[\frac{(1.1 + 1.2)}{1.1}\right](0.1)$$

$$= \frac{6}{5} + \frac{23}{100}$$

$$= \frac{155}{110}$$

$$= \frac{31}{22}.$$

(B)

nce 1440 has prime factorization $2^5 \cdot 3^2 \cdot 5^1$, the number of divi-
$(6)(3)(2) = 36$. This follows by a simple counting scheme in

that divisors of 1440 must have from 0 to 5 factors that are 2, 0 to 2 factors that are 3, and 0 or 1 factors which are 5. The divisor, 1, corresponds to the case when no prime factors are chosen.

62.   (D)

The smallest intersection that is nonempty will contain one point, x, and so we only need count the number of subsets of S that contain x. Each such set will be the union of $\{x\}$ with any subset containing any remaining elements. There are $2^9 = 512$ such subsets. So, the answer is (D).

63.   (D)

Linear combinations provide all solutions in the form $c_1 f(x) + c_2 g(x)$ where c_1, c_2 are constants.

64.   (C)

The characteristic polynomial of A is $|\lambda I - A| = \lambda^2 - 5\lambda - 2$. The Cayley-Hamilton Theorem implies that $A^2 - 5A - 2I = 0$.

65.   (A)

The translation $(x, y) \rightarrow (x, y + 1)$ works for both I and II. The rotation through one radian maps $(\cos n, \sin n)$ into $(\cos(n+1), \sin(n+1))$ leaving the point $(1,0)$ out of the image. No point can map onto the image since n radians cannot equal $2k\pi$ radians as π is irrational.

66.   (B)

An eigenvalue, λ, must satisfy the equation $|A - \lambda I| = 0$. We see that $A - (-2I)$ has all rows equal (all 3's) and so its determinant is immediately zero.

GRE

MATH TEST

TEST V