GRE MATHEMATICS
TEST IV

TIME: 2 hours and 50 minutes
66 Questions

DIRECTIONS: Choose the best answer for each question and mark the letter of your selection on the corresponding answer sheet.

1. Given that (1,2,3) is an eigenvector for the matrix

$$\begin{pmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$, find the corresponding eigenvalue.

(A) 5

(B) 4

(C) 3

(D) 2

(E) None of these

2. A rectangular box resides in 3-space with one vertex at the origin, (0, 0, 0) and three faces in the coordinate planes. If another vertex is $(x, 2, 3)$, $x > 0$, and the angle in radians between the diagonals from (0, 0, 0) to $(x, 2, 3)$ and the other vertex in the xz plane is $\frac{\pi}{6}$, find x.

(A) 1

(B) $\sqrt{2}$

(C) 2

(D) 3

(E) $\sqrt{3}$

3. The random variable, X, is discrete and uniformly distributed with values 1, 2, 3, 4, 5. The variance of X is

(A) 1 (D) 4

(B) 2 (E) None of these

(C) 3

4. Which collection of inequalities represents the shaded region as shown in the plane? The curves represent $y^2 = x - 1$ and $2xy = 1$.

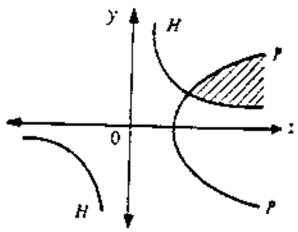
(A) $y^2 + 1 < x$ and $2x > \frac{1}{y}$

(B) $x < y^2 + 1$ or $2x > \frac{1}{y}$

(C) $2x < \frac{1}{y}$ and $x - 1 > y^2$

(D) $2x < \frac{1}{y}$ or $x - 1 < y^2$

(E) $2xy > 1$ and $y^2 + 1 > x$



5. A whispering gallery is constructed as part of the surface formed on rotation of the ellipse $\frac{x^2}{100} + \frac{y^2}{k} = 1$ with x and y in yards. Each whisperer stands at a focus on the x-axis that is three feet from the nearest vertex. Find k.

(A) 6 (B) 7

(C) 18 (E) 36

(D) 19

6. Which of the following is a divisor of $3^{10} - 1$?

(A) 3 (D) 17

(B) 7 (E) 19

(C) 11

7. In the permutation group, S_5, find the product of the elements $(21453) \cdot (35214)$, where $(a\,b\,c\,d\,e)$ means $1 \rightarrow a$, $2 \rightarrow b$, etc.

(A) (41352) (D) (43125)

(B) (43215) (E) (34251)

(C) (43152)

8. In the algebra of sets, which of the following is identical to $(\overline{S} \cap T)$, where $\overline{S}$ denotes the complement of S?

(A) $S \cap \overline{\overline{T}}$ (D) $S \cup \overline{\overline{T}}$

(B) $\overline{(S \cup \overline{\overline{T}})}$ (E) None of these

(C) ϕ

9. Find the interval on which the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^{2n+1}}{n \cdot 4^n}$ converges.

(A) (–1, 7)

(B) (–1, 7]

(C) (1, 5]

(D) (1, 5)

(E) [1, 5]

10. Let f be a function such that the graph of f lies on a parabola through (0, 0) and (4,2). If function g is the composition of f with itself, which of the following points could lie on the graph g?

(A) (1, 1)

(B) (16, 2)

(C) (8, 8)

(D) either (A) or (B)

(E) either (B) or (C)

11. When 4% interest is compounded quarterly, what is the effective annual rate to the nearest hundredth of a percent?

(A) 4.04

(B) 4.03

(C) 4.06

(D) 4.12

(E) 4.08

12. Given that f is a differentiable function with normal line $4x + y = 9$ at $x = 2$ then $\lim_{x \to 0} \frac{f(2-x) - f(2+x)}{x}$ is

(A) $\frac{1}{4}$

(B) $-\frac{1}{4}$

(C) $\frac{1}{2}$

(D) $-\frac{1}{2}$

(E) Undefined

13. Given that a 3×3 matrix A has only one eigenvalue, wh is the dimension of the corresponding eigenspace?

(A) 1

(B) 2

(C) 3

(D) 1 or 2

(E) 1, 2 or 3

14. If $f(e^x) = \sqrt{x}$ for $x \geq 1$, then $f^{-1}(x)$ is

(A) $(\log x)^2$

(B) e^{x^2}

(C) $2 \log x$

(D) $e^{\sqrt{x}}$

(E) $\sqrt{\log x}$

Let $*$ be a binary operation defined on the rational numbers by $a * b = a + b - ab$. Then the number of integers with integer $*$-inverse is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Evaluate the limit, $\lim_{n \to \infty} \sum_{k=1}^{n} \left[1 + \frac{2k}{n}\right]^2 \left[\frac{2}{n}\right]$.

(A) $\frac{26}{3}$

(B) 6

(C) $\frac{13}{3}$

(D) $\frac{7}{3}$

(E) 2

Define a linear transformation from R^2 to R^3 by $(x, y)T = (x + y, x - y, x)$. Define S, a linear transformation from R^3 to R^2, to be an inverse of T if $((x, y)T)S = (x, y)$. Which of the following represent an inverse of T?

(A) $(x,y,z)S = \left(z, \frac{x}{2} - \frac{y}{2}\right)$

(B) $(x,y,z)S = \left(\frac{x}{2} + \frac{y}{2}, \frac{x}{2} - \frac{y}{2} - z\right)$

(C) $(x,y,z)S = \left(\frac{x}{3} + \frac{y}{3} + \frac{z}{3}, x - y\right)$

(D) both (A) and (B)

(E) all three, (A), (B), and (C)

18. The mapping $\phi : G \to G$ given by $g\phi = a^2ga^2$ for fixed $a \in G$ and for each element g in G, is a homomorphism if

(A) $a^4 = e$

(B) $a^3 = e$

(C) $aG = Ga$

(D) G is abelian

(E) G is finite

19. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{pmatrix}$ and $B = A^{-1}$, find the entry in row 3 and column 2 of B.

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) 1

(D) $-\frac{1}{2}$

(E) $-\frac{1}{4}$

20. Given the function $f:(A \cup B) \to C$, which of the following is always true? ($\overline{A}$ denotes the complement of A).

(A) $f(A \cap B) = f(A) \cap f(B)$

(B) $f(\overline{A}) = \overline{f(A)}$

(C) $f(A - B) = f(A) - f(B)$

(D) $f(A \cup B) = f(A) \cup f(B)$

(E) All of these

21. Evaluate $\int_0^{\pi} \cos^2 nx \; dx$, where n is a positive integer.

(A) $\frac{\pi}{2}$

(B) π

(C) $\frac{n\pi}{2}$

(D) $\frac{\pi}{2n}$

(E) $n\pi$

22. Let R be the ring of integers modulo four. Which of the polynomials shown is irreducible over R?

(A) $x^2 + x + 2$

(B) $x^2 + x + 3$

(C) $x^2 + 3x - 2$

(D) $x^3 - 3$

(E) $x^4 + 3$

23. In a circle of radius $\frac{4}{\pi}$ find the area between an arc of length 2 and its chord.

(A) $\frac{4}{\pi}\left[1 - \frac{2}{\pi}\right]$

(B) $\frac{2}{\pi} - \frac{4}{\pi^2}$

(C) $\frac{2}{\pi}$

(D) $\frac{4}{\pi}$

(E) $\frac{8}{\pi}$

24. Let $f_n(x) = x^n$ on $[0, 1]$ and $f(x) = \lim_{n \to \infty} f_n(x)$. Which of the following is true?

(A) f is constant.

(B) f is monotone.

(C) $\{f_n\}$ converges uniformly to f.

(D) Both (A) and (B)

(E) Both (B) and (C)

25. Find $\lim_{x \to \infty} \frac{(e^x + e^{-x})}{2 - 3e^x}$.

(A) 0

(B) $\frac{1}{2}$

(C) Undefined

(D) -2

(E) $-\frac{1}{3}$

26. The column space of a 5×6 matrix is spanned by the vectors $(1,0,0,0,0)$, $(0,0,1,0,0)$, and $(2,0,3,0,0)$. Find the dimension of the solution space of the matrix.

(A) 3

(B) 4

(C) 6

(D) 2

(E) 5

27. For $f(x,y) = x^2 + y^2$, find the rate of change of f at (1,2,5) in the direction $\vec{v} = 3\hat{i} - 4\hat{j}$.

(A) -2

(B) -10

(C) -5

(D) -13

(E) None of these

28. Which pair of the following elements of the symmetric group, S_4, has a product in the alternating group A_4?

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

(A) (i), (ii)

(B) (i), (iii)

(C) (ii), (iii)

(D) both (i), (ii) and (i), (iii)

(E) both (i), (ii) and (ii), (iii)

29. All functions, f, defined on the xy-plane such that

$$\frac{\partial f}{\partial x} = y^2(3x^2 + y^2) \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy(x^2 + 2y^2)$$

are given by

(A) $x^2y^3 + xy^4 + C$

(B) $xy^2(x^2 + y^2) + C$

(C) $x^2y(x^2 + y^2) + C$

(D) $x^4y + x^2y^3 + C$

(E) $xy(x^3 + y^3) + C$

30. The moment with respect to the yz plane of the volume in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ is written as an integral in cylindrical coordinates as $\iiint\limits_V F(z, r, \theta)\, dz\, dr\, d\theta$. Find $F(z, r, \theta)$.

(A) $r \cos\theta$

(B) $r \sin\theta$

(C) $rz \cos\theta$

(D) $r^2 \sin\theta$

(E) $r^2 \cos\theta$

31. In the solution of the differential equation $xy^2\,dx = dy - x\,dx$ satisfying the initial condition $y(0) = -1$, find y when $x = \sqrt{\pi}$

(A) 0

(B) -1

(C) 1

(D) $\frac{\pi}{4}$

(E) $-\frac{\pi}{4}$

32. The set of all points in the plane satisfying $y = x \sin\left(\frac{1}{x}\right)$ together with the origin

(A) is compact but not connected

(B) is connected but not compact

(C) is compact and connected

(D) contains an open set

(E) does not contain all of its limit points.

33. A regular deck of 52 cards contains 4 suits of 13 denominations; 2, 3, …, 10, J, Q, K, A. In how many ways may we select a subset of 5 cards containing exactly 3 of the same denomination? In the answers below, $\cdot$ represents multiplication as usual.

(A) $24 \cdot 47 \cdot 52$ (D) $39 \cdot 47 \cdot 48$

(B) $47 \cdot 48 \cdot 52$ (E) $6 \cdot 47 \cdot 48 \cdot 52$

(C) $24 \cdot 39 \cdot 47$

34. Find the integrating factor for the linear equation $xy' - 2y = x^2$.

(A) x^{-2} (D) x

(B) e^{-2x} (E) x^2

(C) e^{2x}

35. If T is a linear transformation mapping vectors (1,0,0), (0,1,0), and (0,0,1) to the vectors (1,2,3), (2,3,1), and (1,1,–2) respectively, which vector is the image of the vector (3,–2,1) under T?

(A) (1,1,7) (D) (0,1,9)

(B) (1,0,5) (E) (1,7,0)

(C) (0,1,5)

36. If the ninth partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is used to approximate its sum, which statement about E, the sum minus the approximation, is true?

(A) $E > .01$ (D) $-.01 < E < 0$

(B) $0 < E < .01$ (E) $E < -1/81$

(C) $E < -.01$

37. The base ten number, 73, is given in binary as

(A) 10001001 (D) 101001

(B) 1001001

(E) 101011

(C) 10010001

38. For $f(x) = x^3 - 3x^2 + k$ on $[1,3]$, the maximum and minimum values have the same absolute value. Find k.

(A) 4

(D) 1

(B) 3

(E) None of these

(C) 2

39. A regular polygon has ten times as many diagonals as it has vertices. How many sides does it have?

(A) 17

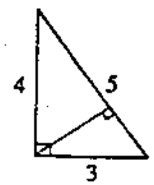
(D) 23

(B) 19

(E) None of these

(C) 21

40. An altitude is drawn to the hypotenuse of the triangle as shown. Find the ratio of the areas of the two smaller triangles.



(A) $\frac{9}{16}$

(D) $\frac{4}{5}$

(B) $\frac{3}{4}$

(E) 1

(C) $\frac{3}{5}$

41. The number of different ways, not counting rotations, to seat 6 different people around a circular table is

(A) 720

(D) 180

(B) 60

(E) 120

(C) 360

42. Find the average value of the function $f(x) = \sqrt{4 - x^2}$ on $[0,2]$.

(A) 1

(D) $\frac{\pi}{2}$

(B) $\frac{3}{2}$

(E) None of these

(C) $\frac{\pi}{4}$

43. Let $F(x)$ represent the Fourier series of the periodic extension of the function $f(x) = x^2$ on the interval, $(0,2)$. Find $F(4)$.

(A) 16

(D) 8

(B) 2

(C) 4

(E) Undefined

44. Let f be a function on $[0,\infty)$ such that for each point in its graph, $(x,y) = (y^2, y)$. At how many points must each such f have a limit?

(A) 1

(B) 2

(C) 3

(D) 4

(E) Infinitely many

45. Given that $\{x_n\}$ is a bounded, divergent, infinite sequence of real numbers, which of the following must be true?

(A) $\{x_n\}$ contains infinitely many convergent subsequences

(B) $\{x_n\}$ contains convergent subsequences with different limits.

(C) $\{y_n = \min_{k \le n} x_k\}$ is convergent.

(D) All of the above

(E) (A) and (C) only

46. Let p, q, r, and s be statements with negations $\overline{p}$, $\overline{q}$, $\overline{r}$, and $\overline{s}$. If p implies q, r implies s, and $\overline{p}$ implies $\overline{q}$ and $\overline{r}$, which statements below must be true?

I. $\overline{p}$ and $\overline{q}$ are equivalent

II. $\overline{r}$ implies $\overline{s}$

III. r implies q

(A) I

(B) II

(C) III

(D) I and II

(E) I and III

47. Given $|x|$, $|y| \le 2$ where x, y are real and $|x| \ne |y|$, we state

I. $|x^2-y^2| \ge 1$ if and only if $|x+y||x-y| \ge 1$,

II. $|x+y||x-y| \ge 1$ if and only if $|x-y| \ge \dfrac{1}{|x+y|}$

III. $|x-y| \ge \dfrac{1}{|x+y|}$ if and only if $|x-y| \ge \dfrac{1}{|x|+|y|}$

IV. $|x-y| \ge \dfrac{1}{|x|+|y|}$ if and only if $|x-y| \ge \dfrac{1}{4}$

Find all the general true statements above.

(A) (I)

(B) (II)

(C) (I) and (II)

(D) (I), (II), and (III)

(E) (I), (II), and (IV)

In a sequence of consecutive throws of a die, find the probability that six will show before a one or a two.

(A) $\frac{1}{6}$ (D) $\frac{5}{6}$

(B) $\frac{1}{2}$ (E) $\frac{1}{3}$

(C) $\frac{2}{3}$

Given $x^2z - 2yz^2 + xy = 0$, find $\frac{\partial x}{\partial z}$ at (1, 1, 1).

(A) 0 (D) 1

(B) $\frac{4}{3}$ (E) None of these

(C) -1

All commutators in a group, G, are of the form $aba^{-1}b^{-1}$. The inverse of this commutator is

(A) the identity element (D) $aba^{-1}b^{-1}$

(B) $a^{-1}b^{-1}ab$ (E) $b^{-1}a^{-1}ba$

(C) $bab^{-1}a^{-1}$

51. If H is a left ideal continuing the identity element of a ring R, then

(A) $RH = H$ (D) Both (B) and (C)

(B) $HR = H$ (E) Both (A) and (C)

(C) $HH = H$

52. Let C_n be a sequence of closed, bounded, nonempty intervals in the real line with the usual topology. The intervals are also nested in the sense that $C_{n+1} \subseteq C_n$. Which of the following is true of the intersection

$$S = \bigcap_{k=1}^{\infty} C_k$$

(A) S may be open or closed.

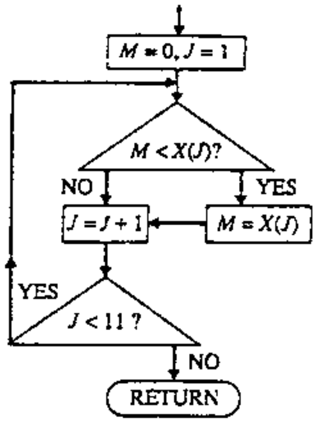
(B) S may be empty.

(C) S must be nonempty and closed.

(D) S must contain an interval.

(E) S must not contain an interval.

53. Find the best statement describing the output that the following subroutine, S, (flow chart) returns to the main program which provides the array $X(I)$.



(A) S returns the minimum of the first 10 values of any $X(I)$

(B) S returns the maximum of the first 11 values of any $X(I)$

(C) S returns the maximum of the first 10 values of any positive $X(J)$

(D) S returns the minimum of the first 11 values of any $X(I)$

(E) S returns the maximum of the first 10 values of any $X(I)$ if one of these values is nonnegative.

54. In drawing two balls, from an urn containing 10 balls of each of the colors, red, white, and blue, find the probability of getting two different colors.

(A) $\frac{2}{3}$

(B) $\frac{1}{3}$

(C) $\frac{10}{29}$

(D) $\frac{20}{29}$

(E) None of these

55. A root of the polynomial $x^3 - 3x^2 + 5x - 3$ is

(A) -1

(B) $2 + \sqrt{2i}$

(C) $1 - \sqrt{8i}$

(D) $1 - \sqrt{2i}$

(E) 3

56. Find the fifth derivative, $f^{(5)}(2)$, of the given

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n(n+1)}$$

(A) 0

(B) $\frac{1}{32}$

(C) $\frac{1}{192}$

(D) $\frac{3}{4}$

(E) $\frac{5}{8}$

57. Let a and b be constants. If $f(x) = ae^{-x^2} + bx\tan x - |x|$ and if $f(7) = 5$, then $f(-7)$ is

(A) 5

(B) -5

(C) -2

(D) Undefined

(E) Not uniquely determined by the information given.

58. If the determinants $|A| = 3$ and $|B| = 2$, find $|2(AB)^{-1}|$ for 4×4 matrices A and B.

(A) $\frac{1}{3}$

(B) $\frac{2}{3}$

(C) $\frac{4}{3}$

(D) $\frac{8}{3}$

(E) 12

59. Let x and y be integers such that $7x + 3y$ is divisible by 13. For which value of k must $8x + ky$ by divisible by 13?

(A) 11

(B) 9

(C) 7

(D) 5

(E) 3

60. In Euler's method of tangents for approximating points on the solution curve of $x\,dy - (x + y)\,dx = 0$, $y(1) = 1$, find the approximation of $y(1.2)$ using two steps of length 0.1.

(A) $\frac{31}{22}$

(B) $\frac{7}{5}$

(C) $\frac{29}{20}$

(D) 2

(E) $\frac{3}{2}$

61. Find the number of distinct divisors of 1440.

(A) 48

(B) 36

(C) 30

(D) 24

(E) 10

62. S is a set containing exactly 10 points. C is a collection of subsets of S. Find the maximum number of subsets that C may contain and satisfy the requirement that the intersection of all the subsets in C is nonempty.

(A) 255

(B) 256

(C) 511

(D) 512

(E) 1023

63. Given that $f(x)$ and $g(x)$ are independent solutions of a linear homogeneous differential equation on (a, b), which of the following must also be solutions?

(A) 0

(B) $2f(x) - 3g(x)$

(C) $f(x)\,g(x)$

(D) Both (A) and (B)

(E) Both (B) and (C)

64. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and let I be an identity matrix. Which matrix polynomial is zero?

(A) $A^2 - 10A + I$

(B) $A^2 - 10A$

(C) $A^2 - 5A - 2I$

(D) $A^2 + 5A - 2I$

(E) $A^2 + 5A + 2I$

65. Which of the following subsets of the plane may be translated or rotated into a proper subset of itself?

I. $Z = \{(x, y) \mid x$ and y are positive integers$\}$

II. $I = \{(x, y) \mid y \geq x\}$

III. $C = \{(\cos n, \sin n) \mid n = 0,1,2,\ldots\}$

(A) I and II

(B) I and III

(C) II and III

(D) I, II, and III

(E) None of these

66. Which of the following is an eigenvalue of the matrix

$$A = \begin{pmatrix} 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{pmatrix}$$

(A) -1

(B) -2

(C) 1

(D) 2

(E) 0

GRE MATHEMATICS TEST IV

ANSWER KEY

A	23.	A	45.	D
E	24.	B	46.	E
B	25.	E	47.	C
A	26.	B	48.	D
D	27.	A	49.	D
C	28.	C	50.	C
D	29.	B	51.	E
D	30.	E	52.	C
E	31.	C	53.	E
E	32.	B	54.	D
C	33.	A	55.	D
D	34.	A	56.	E
E	35.	C	57.	A
B	36.	B	58.	D
C	37.	B	59.	B
A	38.	C	60.	A
A	39.	D	61.	B
A	40.	A	62.	D
E	41.	E	63.	D
D	42.	D	64.	C
A	43.	D	65.	A
B	44.	A	66.	B

GRE MATHEMATICS TEST IV

DETAILED EXPLANATIONS OF ANSWERS

1. (A)

When $\overline{x}$ is an eigenvector for matrix, A, then $A\overline{x} = \lambda\overline{x}$ defines the eigenvalue, λ. In this case,

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}5\\10\\15\end{pmatrix} = 5\begin{pmatrix}1\\2\\3\end{pmatrix} \text{ so } \lambda = 5.$$

2. (E)

The other vertex is $(x, 0, 3)$ and so, using dot products,

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{3}{4}}$$

$$= \frac{x^2+9}{\sqrt{x^2+13}\sqrt{x^2+9}}$$

$$= \sqrt{\frac{x^2+9}{x^2+13}}$$

Then $4(x^2+9) = 3(x^2+13)$ implies $x^2 = 3$ and $x = \sqrt{3}$.