GRE MATHEMATICS TEST III

DETAILED EXPLANATIONS OF ANSWERS

1. (B)

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence. The generating function for this sequence is given by

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots .$$

Since the Fibonacci sequence satisfies $a_n = a_{n-1} + a_{n-2}$ for $n = 2, 3, 4, \ldots$, we have

$$f(x) = a_0 + a_1x + (a_0 + a_1)x^2 + (a_1 + a_2)x^3 + \ldots$$
$$+ (a_{n-1} + a_{n-2})x^n + \ldots$$
$$= a_0 + x(a_1 + a_1x + a_2x^2 + \ldots) + x^2(a_0 + a_1x + \ldots)$$
$$= 1 + x f(x) + x^2 f(x)$$

where use was made of the fact that $a_0 = a_1 = 1$. Hence

$$(1 - x - x^2) f(x) = 1$$

so that

$$f(x) = (1 - x - x^2)^{-1}$$

2. (E)

We have

$$p(0) = 0 + 0 + 2 = 2 \equiv 2 \pmod 6$$
$$p(1) = 1 + 3 + 2 = 6 \equiv 0 \pmod 6$$
$$p(2) = 4 + 6 + 2 = 12 \equiv 0 \pmod 6$$
$$p(3) = 9 + 9 + 2 = 20 \equiv 2 \pmod 6$$
$$p(4) = 16 + 12 + 2 = 30 \equiv 0 \pmod 6$$
$$p(5) = 25 + 15 + 2 = 42 \equiv 0 \pmod 6$$

so that there are four elements of Z_6 satisfying $p(x) = 0$.

3. (D)

A complex matrix M is said to be normal if $M M^* = M^* M$, where M^* is the conjugate transpose of M. If M is real then $M^* = M^T$, where M^T is the transpose of M. We have for

$$M = \begin{bmatrix} i & 1 \\ -1 & 0 \end{bmatrix}, \quad M^* = \begin{bmatrix} -i & -1 \\ 1 & 0 \end{bmatrix}$$

so that

$$M M^* = \begin{bmatrix} i & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$$

and

$$M^* M = \begin{bmatrix} -i & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix},$$

so M is normal.

4. (D)

We have

$$\sqrt{x^2 + (y-1)^2} + \sqrt{(x-1)^2 + y^2} = 2$$
$$\sqrt{x^2 + (y-1)^2} = 2 - \sqrt{(x-1)^2 + y^2}$$

$$4\sqrt{(x-1)^2+y^2}=4-2x+2y$$
$$3x^2+2xy+3y^2-4x-4y=0$$

5. (A)

Let $\{u_k\}_k^{+\infty}=1$ be a sequence of non-zero numbers and form the sequence $\{p_n\}_n^{+\infty}=1$ of partial products $p_n = u_1 \cdot u_2 \dots u_n = \prod_{k=1}^{n} uk$ for $n = 1, 2, \dots$. (actually, this sequence does not have to start at $n = 1$). If p_n converges to p, we say that the infinite product $\prod_{k=1}^{+\infty} u_k$ converges to p. For $u_k = 1 - \frac{1}{k^2}$ we have

$$p_2 = \frac{3}{4} \qquad \left(= \frac{2+1}{2\cdot 2}\right)$$
$$p_3 = \frac{3}{4}\,\frac{8}{9} = \frac{2}{3} \qquad \left(= \frac{3+1}{2\cdot 3}\right)$$
$$p_4 = \frac{2}{3}\,\frac{15}{16} = \frac{5}{8} \qquad \left(= \frac{4+1}{2\cdot 4}\right)$$
$$p_5 = \frac{5}{8}\,\frac{24}{25} = \frac{3}{5} \qquad \left(= \frac{5+1}{2\cdot 5}\right)$$

We claim that $p_n = \frac{n+1}{2n}$. This is true for $p_2 = \frac{2+1}{4} = \frac{3}{4}$.

We assume that it is true for $m = n$ and let $m = n + 1$. Then

$$p_n + 1 = p_n u_{n+1}$$
$$= \left[\frac{n+1}{2n}\right]\left[\frac{(n+1)^2-1}{(n+1)^2}\right]$$
$$= \frac{n^2+2n}{2n\,(n+1)}$$
$$= \frac{(n+1)+1}{2\,(n+1)}$$

Therefore $p_n = \frac{n+1}{2n}$ for $n = 2, 3, \dots$ so that

$$\prod_{k=2}^{+\infty}\left(1-\frac{1}{k^2}\right) = \lim_{n\to+\infty} p_n$$
$$= \lim_{n\to+\infty} \frac{n+1}{2n}$$
$$= \frac{1}{2}$$

6. (A)

The cross ratio R_c of a set of four distinct concurrent lines l_1, l_2, l_3, l_4, is given by

$$R_c = \frac{(m_3-m_1)\,(m_4-m_2)}{(m_3-m_2)\,(m_4-m_1)}$$

where m_1, m_2, m_3, and m_4 represent the slopes of l_1, l_2, l_3 and l_4 respectively. We have $m_1 = 1/2$, $m_2 = 2/2$, $m_3 = 3/2$, and $m_4 = 4/2$ so that

$$R_c = \frac{(3-1)\,(4-2)}{(3-2)\,(4-1)} = \frac{4}{3}$$

7. (B)

Let R be a ring and n a positive integer such that $n \cdot r = 0$ for all $r \in R$. Then the least positive integer satisfying the equation is called the characteristic of R. If there does not exist a positive integer satisfying the equation, then R is said to have characteristic 0. The ring

$$Z_2 + Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

We have

$$1(0,1) \neq (0,0)$$
$$2(0,1) = (0,2) \neq (0,0)$$

$3(1,1) = (1, 0) \neq (0, 0)$
$4(1,1) = (0, 1) \neq (0, 0)$
$5(0,1) = (0, 2) \neq (0, 0)$

However, $6(r,\overline{r}) = (0,0)$ for all $(r,\overline{r}) \in Z_2 + Z_3$, so the characteristic of $Z_2 + Z_3$ is 6.

8. (A)

We have

$$\lim_{n\to+\infty} \left(\sqrt{n^4 + in^2} - n^2\right)$$

$$= \lim_{n\to+\infty} \left(\sqrt{n^4 + in^2} - n^2\right) \frac{\sqrt{n^4 + in^2} + n^2}{\sqrt{n^4 + in^2} + n^2}$$

$$= \lim_{n\to+\infty} \frac{in^2}{\sqrt{n^4 + in^2} + n^2}$$

$$= \lim_{n\to+\infty} \frac{i}{\sqrt{1 + \frac{i}{n^2}} + 1}$$

$$= \frac{i}{2}$$

9. (A)

Successive substitutions of the right hand side of

$$u(x) = x + \int_0^x (t - x)\, u(t)\, dt$$

into the $u(t)$ in the integral yields

$$u(x) = x + \int_0^x (t - x) \left[t + \int_0^t (t_1 - t)\, u(t_1)\, dt_1 \right] dt$$

$$= x + \int_0^x (t^2 - xt)\, dt + \int_0^x (t - x) \int_0^t (t_1 - t)\, u(t_1)\, dt_1\, dt$$

$$= x - \frac{x^3}{3!} + \int_0^x (t - x) \int_0^t (t_1 - t)\, u(t_1)\, dt_1\, dt$$

$$= x - \frac{x^3}{3!} + \int_0^x (t - x) \int_0^t (t_1 - t) \left[t_1 + \int_0^{t_1} (t_2 - t_1)\, u(t_2) dt_2 \right] dt_1 dt$$

$$= x - \frac{x^3}{3!} + \int_0^x (t - x) \int_0^t (t_1 - t)\, u(t_1)\, dt_1\, dt$$

$$+ \int_0^x (t - x) \int_0^t (t_1 - t) \int_0^{t_1} (t_2 - t_1)\, u(t_2)\, dt_2\, dt_1\, dt$$

$$= x - \frac{x^3}{3!} + \int_0^x (t - x) \left[-\frac{t^3}{3!} \right] dt$$

$$+ \int_0^x (t - x) \int_0^t (t_1 - t) \int_0^{t_1} (t_2 - t_1)\, u(t_2)\, dt_2\, dt_1\, dt$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \int_0^x (t - x) \int_0^t (t_1 - t) \int_0^{t_1} (t_2 - t_1)\, u(t_2)\, dt_2\, dt_1$$

$$= \ldots = \sin x$$

10. (A)

Setting $u = \ln x^{-1}$, we obtain $x = e^{-u}$ and $dx = -e^{-u}\, du$. Also, $x \to 0^+$ implies $u \to +\infty$ and $x \to 1^-$ implies $u \to 0^+$. Thus

$$\int_0^1 \left(\ln \frac{1}{x}\right)^5 dx = \int_{+\infty}^0 u^5 \left(-e^{-u}\right) du$$

$$= \int_0^{+\infty} u^{6-1}\, e^{-u}\, du$$

$$= \Gamma(6) = 5! = 120$$

11. (E)

Let $f(x)$ be defined on $[0, +\infty)$. The Laplace transform of $f(x)$, denoted $\pounds[f(x)](p)$, is the function of p defined by

$$\pounds\,[f(x)](p) = \int_0^{+\infty} e^{-px} f(x)\; dx$$

The domain of $\pounds[f(x)](p)$ is the set of all real numbers p for which the integral converges. We have

$$\pounds[f(x)](p) = \int_1^{+\infty} e^{-px}\; dx$$

$$= \lim_{b \to +\infty} \int_1^b e^{-px}\; dx$$

$$= \lim_{b \to +\infty} \left[-\frac{e^{-px}}{p} \right]_1^b$$

$$= \lim_{b \to +\infty} \left[\frac{e^{-p}}{p} - \frac{e^{-pb}}{p} \right]$$

$$= \frac{e^{-p}}{p}$$

for $p \,\varepsilon\, (0, +\infty)$.

12. (A)

The matrix M_T of T relative to $\{(1, 0), (0, 1)\}$ is given by

$$M_T = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

The matrix $M_T{}^*$ for the adjoint T^* of T is

$$M_T{}^* = M_T^* = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Thus

$$T^*(x, y) = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ -x + 3y \end{bmatrix}$$

13. (D)

Set $x = \pi/2 - u$. Then $dx = -du$, $x = 0$ implies that $u = \pi/2$, and $x = \pi/2$ implies that $u = 0$. Using the identities $\sin(\pi/2 - u) = \cos u$ and $\cos(\pi/2 - u) = \sin u$, we obtain

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\, dx = \int_{\frac{\pi}{2}}^0 \frac{\sin u}{\sin u + \cos u}\,(-du)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin u}{\sin u + \cos u}\, du$$

Hence

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} 1\; dx$$

$$= \frac{\pi}{2}$$

so that $I = \pi/4$.

14. (D)

The discriminant D of the ternary quadratic form

$$a_{11} x^2 + a_{22} y^2 + a_{33} z^2 + 2\, a_{12}\, xy + 2\, a_{23}\, yz + 2\, a_{13}\, xz$$

is the determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Thus

$$D = \det \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$$

$$= (-1 + 6 + 6) - (-9 + 4 + 1)$$

$$= 15$$

15. (E)

The curvature $\kappa(x)$ of $f(x)$ at $P(x_0, f(x_0))$ is given by

$$\kappa(x_0) = \frac{f''(x_0)}{\{1 + [f'(x_0)]^2\}^{3/2}}$$

and the radius of curvature $R(x)$ of $f(x)$ at $P(x_0, f(x_0))$ is given by

$$R(x_0) = \frac{1}{|\kappa(x_0)|}.$$

We have

$$f'(x) = 1 - \frac{1}{x^2}\,;\; f'(1) = 0$$

$$f''(x) = -\frac{2}{x^3};\quad f''(1) = -2$$

so that $\kappa(1) = -2$ and $R(1) = 1/2$.

16. (B)

The gamma function $\Gamma(p)$ is defined by

$$\Gamma(p) = \int_0^{+\infty} x^{p-1} e^{-x}\, dx$$

Setting $x = u^{1/2}$, we obtain $dx = u^{1/2}\, du$ so that

$$\int_0^{+\infty} e^{-x^2}\, dx = \int_0^{+\infty} \frac{1}{2}\, u^{-1/2} e^{-u}\, du$$

$$= \frac{1}{2} \int_0^{+\infty} u^{1/2-1} e^{-u}\, du$$

$$= \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$$

17. (B)

We have

$$<(1, 0)> = \{(0, 0), (1, 0)\}$$

so that all cosets of $<(1, 0)>$ must have two elements. Since $Z_2 \times Z_3$ h
six elements, $(Z_2 \times Z_3)/<(1, 0)>$ contains three elements:

$$(0, 1) + <(1, 0)> = \{(0, 1), (1, 1)\}$$
$$(0, 0) + <(1, 0)> = \{(0, 0), (1, 0)\}$$
$$(1, 2) + <(1, 0)> = \{(1, 2), (0, 2)\}$$

18. (C)

The function d would be called a metric for $R[0, 1]$ if it satisfied
of the conditions (A) – (E). Since condition (C) is not satisfied, t
function d is called a pseudometric for $R[0, 1]$. To see this, define

$$f(x) = 1 \text{ if } x \in [0, 1]$$

and

$$g(x) = \begin{cases} 1 \text{ if } x \in [0, 1) \\ 2 \text{ if } x = 1 \end{cases}$$

Then $f, g \in R[0, 1]$ with $f \neq g$, but $d(f, g) = 0$, as

$$\int_0^1 |f(x) - g(x)|\, dx = \int_0^1 |1 - 1|\, dx = 0 .$$

19. (E)

Let m (> 0) and n be integers. The Division Algorithm states that there exist unique integers q and r such that

$$\frac{n}{m} = q + \frac{r}{m} \text{ where } 0 \leq \frac{r}{m} < 1.$$

Repeated applications of this algorithm yield

$$\frac{13}{42} = 0 + \cfrac{1}{\cfrac{42}{13}} = \cfrac{1}{3 + \cfrac{1}{\cfrac{13}{3}}} = \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{3}}} .$$

20. (E)

Let $p(x) = a_N x^N + a_{N-1} x^{N-1} + \ldots + a_1 x + a_0$ be an element of $Z[x]$. The Eisenstein criterion states that for a prime p, if $a_N \not\equiv 0 \pmod p$, $a_j \equiv 0 \pmod p$ for $j = 0, 1, 2, \ldots, N-1$ and $a_0 \not\equiv 0 \pmod{p^2}$, then $p(x)$ is irreducible over the rationals. We have, for $p = 3$,

$$5 \not\equiv 0 \pmod 3$$
$$9 \equiv 0 \pmod 3$$
$$0 \equiv 0 \pmod 3$$
$$18 \equiv 0 \pmod 3$$
$$3 \equiv 0 \pmod 3$$
$$15 \equiv 0 \pmod 3$$
$$15 \not\equiv 0 \pmod 9$$

Therefore $5x^5 + 9x^4 + 18x^2 + 3x + 15$ satisfies an Eisenstein criterion $(p = 3)$.

21. (B)

The angle θ that a conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ must be rotated in order to eliminate the xy term must satisfy

$$\cot 2\theta = \frac{A - C}{B} .$$

For the given conic, we must have

$$\cot 2\theta = \frac{1 + 1}{2\sqrt{3}} \text{ or } \tan 2\theta = \sqrt{3} .$$

Therefore $\theta = 30^\circ$.

22. (D)

We have

$$\begin{aligned} 0 &= \pounds\,[y'' + 6y' + 9y] \\ &= p^2 \pounds(y) - [p\, y(0) + y'(0)] + 6\{p\, \pounds(y) - y(0)\} + 9\, \pounds(y) \\ &= [p^2 + 6p + 9]\, \pounds(y) = 3p + 7 \end{aligned}$$

Thus

$$\pounds(y) = \frac{3p+7}{(p+3)^2}$$

$$= \frac{A}{p+3} + \frac{B}{(p+3)^2}$$

Setting $p = -3$ in $3p + 7 = A(p + 3) + B$ yields $B = -2$. For $p = 0$, we obtain $A = 3$.

23. (B)

The join of subgroups S_1 and S_2 of G is the smallest subgroup of G containing S_1 and S_2. Thus the join must contain $\{0, 4, 6, 8\}$. Since $4 + 6 \equiv 10 \pmod{12}$ and $6 + 8 \equiv 2 \pmod{12}$, the join equals <2>.

24. (B)

Since a 60° angle cannot be trisected, a 20° angle cannot be constructed. A regular 9–gon is constructible if and only if the angle $\frac{360}{9} = 40°$ is constructible. Thus, a 9–gon is not constructible because one of the 40° angles formed could then be bisected to construct a 20° angle.

25. (E)

We have

$$T^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

and, in general,

$$T^n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}.$$

Hence, the sum of the elements in T^n is $n + 2$.

26. (D)

Let V be a finite dimensional vector space over a field F. A bilinear form $b(v;\overline{v})$ is a function $b: V \times V \rightarrow F$ which satisfies

1) $b(\alpha v_1 + \beta v_2 ; \overline{v}) = \alpha b(v_1 ; \overline{v}) + \beta\, b(v_2 ; \overline{v})$

2) $b(v ; \delta\overline{v}_1 \upsilon\overline{v}_2) = \delta(v ; \overline{v}_1) + \upsilon\, b(v ; \overline{v}_2)$

where $v_1, v_2, \overline{v}_1, \overline{v}_2 \, \varepsilon \, V$ and α, β, δ, υ, ε F. The matrix $B = (b_{ij})$ is given by $b_{ij} = b(u_i, u_j)$ for $1 \le i \le 2$; $1 \le j \le 2$. We have

$$b_{11}(u_1, u_1) = b((0, 1); (0, 1)) = 0 + 0 + 0 + 0 = 0$$
$$b_{12}(u_1, u_2) = b((0, 1); (1, 1)) = 0 + 0 + 1 + 3 = 4$$
$$b_{21}(u_2, u_1) = b((1, 1); (0, 1)) = 1 - 2 + 0 + 0 = -1$$
$$b_{22}(u_2, u_2) = b((1, 1); (1, 1)) = 1 - 2 + 1 + 3 = 3$$

Thus

$$B = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}$$

27. (D)

A class of subsets τ of $\mathcal{X}$ is called a topology on $\mathcal{X}$ if τ satisfies the following axioms:

(1) $X \in \tau$; $\varnothing \in \tau$

(2) $S_\alpha \in \tau$ for $\alpha \in A$ implies $\bigcup_{\alpha \in A} S_\alpha \in \tau$

(3) $S_1, S_2 \in \tau$ implies $S_1 \cap S_2 \in \tau$

Consider $\tau = \{X, \varnothing, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. We have $\{a, b\} \in \tau$ and $\{b, c\} \in \tau$ but $\{b\} = \{a, b\} \cap \{b, c\}$ is not an element of τ. Thus τ does not form a topology on X.

28. (B)

An eigenvalue λ for a differential equation of the form $y'' + \lambda y = 0$ with initial conditions $y(x_1) = y_1$; $y(x_2) = y_2$ is a number for which the initial value problem has a non-trivial solution. The auxiliary equation of the given equation is $m^2 + \lambda = 0$ so that $m = \pm\sqrt{-\lambda}$. We consider three cases: $\lambda = 0$; $\lambda < 0$; $\lambda > 0$. For $\lambda = 0$, the general solution is $y(x) = c_1 + c_2 x$. Since $y(0) = 0$ and $y(\pi) = 0$, we have

$$0 = y(0) = c_1$$

$$0 = y(\pi) = c_2\pi$$

so that $y(x) \equiv 0$. For $\lambda < 0$, the general solution is

$$y(x) = c_1 e\sqrt{-\lambda}x + c_2 e - \sqrt{-\lambda}x.$$

Substitution of the initial conditions into $y(x)$ yields

$$0 = y(0) = c_1 + c_2$$

$$0 = c_1 e\sqrt{-\lambda}\pi + c_2 e - \sqrt{-\lambda}\pi \ .$$

Since

$$\begin{vmatrix} 1 & 1 \\ e\sqrt{-\lambda}\pi & e - \sqrt{-\lambda}\pi \end{vmatrix} \neq 0$$

$c_1 = c_2 = 0$, so that $y(x) \equiv 0$. For $\lambda > 0$, the general solution is

$$y(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x \ .$$

Substitution of the initial conditions into $y(x)$ yields

$$0 = y(0) = c_2$$

$$0 = y(\pi) = c_1 \sin\sqrt{\lambda}\ x \ .$$

To obtain a non-trivial solution, $\sin\sqrt{\lambda}x = 0$, that is $\sqrt{\lambda}\pi = k\pi$ for $k = 1, 2, 3, \ldots$. Therefore, the eigenvalues are $\lambda = k^2$ for $k = 1, 2, 3, \ldots$.

29. (D)

Switching algebra is similar to set algebra with union ($\cup$) replaced with disjunction ($\vee$), intersection replaced with conjunction (or juxtaposition) and complementation denoted by ($\bar{\ }$) . We have

$$(f \vee g)(\bar{f} \vee h) = (f \vee g)\bar{f} \vee (f \vee g)h$$

$$= f\bar{f} \vee g\bar{f} \vee fh \vee gh$$

$$= 0 \vee g\bar{f} \vee fh \vee gh$$

$$= g\bar{f} \vee fh \vee gh$$

30. (C)

Using the identity $\|T\|^2 = \langle T, T\rangle$, we obtain

$$\|T\| = \text{trace}\ (T^t T)$$

$$= \text{trace}\left[\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\right]$$

$$= \text{trace}\begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$$

$$= 15$$

31. (E)

A geometric series is a series of the form $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$. If $|r| < 1$, the series converges to $\frac{a}{1-r}$. For the given series, $r = \frac{1}{1+x^2} < 1$ so that its sum is

$$\frac{x^4}{1 - \frac{1}{1+x^2}} = \frac{x^4}{\frac{1+x^2-1}{1+x^2}} = x^4 + x^2 .$$

32. (A)

To within an additive constant,

$$f(x) = \frac{x^2}{2} + \frac{2x^3}{3} + \ldots + \frac{n\,x^{n+1}}{n+1} + \ldots$$

Using the ratio test with $u_n = \frac{n\,x^{n+1}}{n+1}$, we have

$$L = \lim_{n \to +\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n \to +\infty} \left| \frac{(n+1)\,(n+1)\,x\,x^{n+1}}{n\,(n+2)\,x^{n+1}} \right|$$

$$= |x|$$

so that the series is absolutely convergent for $|x| < 1$ and divergent for $|x| > 1$. For

$$x = -1,\ u_n = \frac{(-1)^{n+1}\,n}{n+1}$$

which does not go to 0 which implies that -1 is not in Dom(f). For

$$x = 1,\ u_n = \frac{n}{n+1},$$

which does not approach 0 which imples that 1 is not in Dom(f). Thus Dom(f) = $(-1, 1)$.

33. (C)

Let $(G, *)$ and $(G', *')$ be groups. A homomorphism from G into G' is a function $\varnothing$ such that $\varnothing(gh) = \varnothing(g)\,\varnothing(h)$. A homomorphism $\varnothing$ must take the identity $e \in G$ onto the identity $\varnothing(e) \in G'$. When G is cyclic, the homomorphism $\varnothing$ is completely determined by $\varnothing(g)$ where g is a generator of G. Since 1 is a generator of Z_8, the value $\varnothing(1) \in \{0, 1, 2, 3\}$ is critical. Because we desire only the "onto" homomorphisms, $\varnothing(1)$ must be a generator of Z_4. The generators of Z_4 are the elements of $\{0, 1, 2, 3\}$ relatively prime to 4, namely 1 and 3. The functions "defined by" $\varnothing_1(1) = 1$ and $\varnothing_2(1) = 3$ are both homomorphisms from Z_8 onto Z_4. In particular, note that they both take the identity element in Z_8 onto the identity element in Z_4.

34. (C)

Setting $x = e^t$, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'\,x, \text{ since } \frac{dx}{dt} = \frac{d(e^t)}{dt} = e^t = x$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = x\,\frac{d}{dt}\left(\frac{dy}{dx}\right) + xy'$$

$$= x\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + xy' = x^2y'' + xy'$$

Thus

$$xy' = \frac{dy}{dt} \text{ and } x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

We have

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 6\,\frac{dy}{dt} + 6y = 0$$

$$\frac{d^2y}{dt^2} + 5\,\frac{dy}{dt} + 6y = 0 .$$

The auxiliary equation is given by $m^2 + 5m + 6 = 0$, so that the general solution is $c_1 e^{-3t} + c_2 e^{-2t}$ which equals $\frac{c_1}{x^3} + \frac{c_2}{x^2}$.

35. (E)

Fermat's "little" theorem states that if t is an integer and p a prime not dividing t, then $t^{p-1} \equiv 1 \pmod p$. Thus

$$5^{34} \equiv (5^{16})^2\, 5^2 \equiv (5^{17-1})^2\, 5^2 \equiv 1^2\, 5^2 \equiv 25 \equiv 8 \pmod{17}$$

36. (C)

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. The curl of $\vec{u}$, denoted curl $(\vec{u})$, is the cross product of the vector operator $\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ and the vector $\vec{u}$:

$$\text{curl}(\vec{u}) = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

For $\vec{u} = xyz\,\vec{i} + xy^2\,\vec{j} + yz\,\vec{k}$, we have

$$\text{curl}(\vec{u}) = (z - 0)\,\vec{i} + (xy - 0)\,\vec{j} + (y^2 - xz)\,\vec{k}$$

37. (B)

The inverse f^{-1} of f is found from $y = \frac{x}{x-1}$ by first solving for x in terms of y and then replacing y with x and x with y. Thus

$$y = \frac{x}{x-1}$$

$$xy - y = x$$

$$x(y-1) = y \Rightarrow x = \frac{y}{y-1}$$

$$f^{-1}(x) = \frac{x}{x-1}$$

Thus $f^{-1}(x) = f(x)$.

38. (B)

For $x_0 = -2$, we have $8 + y_0^2 + 30 = 8y_0 + 24$ so that

$$y_0 = \frac{8 \pm \sqrt{64 - 4(14)}}{2} = 4 \pm \sqrt{2}$$

Since $y_0 > 4$, $y_0 = 4 \pm \sqrt{2}$. Taking the derivative of both sides of $2x^2 + y^2 + 30 = 8y - 12x$ with respect to x yields

$$4x + 2yy' = 8y' - 12$$

$$4x + 12 = y'(8 - 2y)$$

$$y' = \frac{2x + 6}{4 - y}$$

Therefore, the slope of the tangent line at $(-2, 4 \pm \sqrt{2})$ is

$$y' = \frac{-4 + 6}{4 - (4 + \sqrt{2})} = -\sqrt{2}$$

39. (B)

The equation $D = aA + bB + cC$ implies that

$$\begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} a - c & a + b \\ b + c & -a + b + c \end{bmatrix}$$

Thus,

$$a - c = 3$$

$$a + b = 3$$

$$b + c = 0$$

$$-a + b + c = -2$$

Since $b + c = 0$, $a = 2$ which implies that $b = 1$, $c = -1$.

40. (B)

The derivative of $p(x)$ is given by

$$p'(x) = \sum_{n=1}^{+\infty} \frac{(x-2)^{n-1}}{n}$$

The nth term of $p'(x)$ is $a_n = \dfrac{(x-2)^{n-1}}{n}$.

$$\rho = \lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(x-2)^n n}{(x-2)^{n-1}(n+1)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(x-2)\,n}{n+1}\right|$$
$$= |x-2|$$

By the Ratio Test, the series converges absolutely if $|x-2|<1$, i.e., $1<x<3$. Now we consider endpoints. When $x=1$, the series converges by the Alternating Series Test. When $x=3$, the series is the divergent harmonic series. Therefore, the series converges in $[1, 3]$.

41. (C)

Let $f(x)$ be defined on $[0, +\infty)$. The Laplace transformation of f, denoted $\pounds[f(x)](p)$, is defined by

$$\pounds[f(x)](p) = \int_0^{+\infty} e^{-px} f(x)\, dx$$

for all p for which the integral converges. If, in addition, f is of exponential order and is piecewise continuous on $[0, b]$ for every $b>0$, then

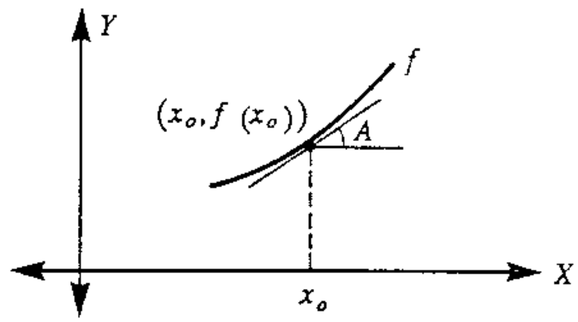
$$\pounds\left[\int_0^x f(t)\,dt\right](p) = \frac{1}{p}\pounds[f(x)](p)$$

Since $\pounds[\sin ax](p) = \dfrac{a}{p^2+a^2}$, we have

$$\pounds\left[\int_0^x \sin 2t\,dt\right](p) = \frac{2}{p^3+4p}.$$

42. (D)

The slope of the tangent line to the graph of f at x_0 is equal to $f'(x_0)$. It is also equal to the tan A where A is the angle formed by the tangent line and the positive x-axis. Thus $\tan A = f'(x_0) = \sqrt{3}$ so that $a = 60°$.



43. (E)

The trace of T is the trace of M_T where M_T is a matrix representing T in some fixed basis. For the basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

$$M_T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

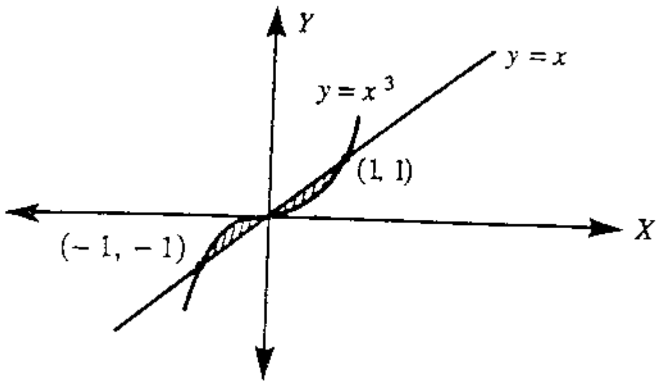
so that the trace of T is 2.

44. (B)

The n'th partial sum s_n can be written as

$$s_n = \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}$$
$$= \frac{2-1}{2!}+\frac{3-1}{3!}+\frac{4-1}{4!}+\ldots+\frac{(n+1)-1}{(n+1)!}$$
$$= 1-\frac{1}{2!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}-\frac{1}{4!}+\ldots+\frac{1}{n!}-\frac{1}{(n+1)!}$$
$$= 1-\frac{1}{(n+1)!}\to 1 \text{ as } n\to+\infty.$$

Therefore, the sum is 1.



45. (A)

An ideal I of a ring R is a subring of R such that $rI \subseteq I$ and $Ir \subseteq I$, for all $r \in R$. An ideal I is called proper if $I \neq R$ and $I \neq \{0\}$. Since 5 and 12 are relatively prime, 5 is a generator of Z_{12}, that is, $<5> = Z_{12}$. Thus, $<5>$ is not a proper ideal of Z_{12}.

46. (A)

A set of answers to the ten questions is a 10-tuple of the form $(a_1, \ldots, a_{10})$ where a_j represents an answer to question Q_j, $1 \leq j \leq 10$. An answer is an element of the set $\{r_j, w_{j1}, \ldots, w_{j4}\}$ where r_j represents the right answer and $w_{j1}, \ldots, w_{j4}$ represent wrong answers. There are 5^{10} possible sets of answers. The number of ways to obtain a 10-tuple with exactly five correct answers is

$$\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{10!}{5!\,5!} = 252,$$

each with probability

$$\left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^5 = \frac{4^5}{5^{10}}.$$

Hence the probability desired is

$$\frac{252\ 4^5}{5^{10}} = \frac{(63)\ 4^6}{5^{10}}.$$

47. (D)

The Jacobian J of a transformation from the xy-plane into the uv- plane defined by

$$u = f(x, y)$$

$$v = g(x, y)$$

is given by

$$J = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

We have

$$J = \begin{vmatrix} xye^{xy} + e^{xy} & x^2 e^{xy} \\ y^2 e^{xy} & xye^{xy} + e^{xy} \end{vmatrix}$$

$$= e^{2xy}\left[(xy + 1)^2 - x^2 y^2\right]$$

$$= [2xy + 1]\ e^{2xy}.$$

48. (A)

The outcome of six attempts is a 6-tuple of the form $(O_1, \ldots, O_6)$, where O_j is either a hit (H) or an out (O). There are $2^6 = 64$ possible outcomes. The number of ways to select a 6-tuple with exactly 3 H's and 3 O's is

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \frac{6!}{3!\,3!} = 20,$$

each with probability

$$\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{8}{3^6}.$$

Hence the desired probability is $\frac{160}{3^6}$.

49. (E)

Let R be a ring with a multiplicative identity e ($re = er = r$, for all $r \varepsilon R$). A unit of R is an element u of R that possesses a multiplicative inverse in R. In $Z_5 = \{0, 1, 2, 3, 4\}$, we have that 1 is the multiplicative identity and

$$1 \quad 1 \equiv 1 \pmod 5$$

$$2 \cdot 3 \equiv 1 \pmod 5$$

$$3 \cdot 3 \equiv 1 \pmod 5$$

$$4 \cdot 4 \equiv 1 \pmod 5$$

which means that Z_5 contains four units.

50. (A)

Let f be defined on an interval $(0, L)$. Define f in $(-L, L)$ to be odd. The half range Fourier sine series is given by

$$\sum_{k=1}^{+\infty} b_k \sin \frac{k\pi x}{L}$$

where

$$b_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L}\, dx$$

for $k = 1, 2, 3, \ldots$. We have

$$b_k = 2 \int_0^1 x \sin k\pi x \, dx$$

$$= 2 \left\{ \left[-\frac{x}{k\pi} \cos k\pi x \right]_0^1 + \int_0^1 \frac{1}{k\pi} \cos k\pi x \, dx \right\}$$

$$= 2 \left\{ -\frac{1}{k\pi} \cos k\pi + \frac{1}{k\pi} \left[\frac{1}{k\pi} \sin k\pi x \right]_0^1 \right\}$$

$$= -\frac{2}{k\pi} \cos k\pi \, .$$

For $k = 3$, $b_3 = -\frac{2}{3\pi} \cos 3\pi = \frac{2}{3\pi}$.

51. (C)

Since $\{e^x, e^{2x}, e^{-2x}\}$ is a linearly independent set, it forms a b
for S. In this basis, we have

$$D_x(e^x) = 1e^x + 0e^{2x} + 0e^{-2x}$$

$$D_x(e^{2x}) = 0e^x + 2e^{2x} + 0e^{-2x}$$

$$D_x(e^{-2x}) = 0e^x + 0e^{2x} - 2e^{-2x}$$

so that the corresponding matrix M_{D_x} of D_x is given by

$$M_{D_x} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The determinant of D_x is -4.

52. (B)

Let $a_2(x), a_1(x), a_0(x)$, and $f(x)$ be continuous on an interval I w
$a_2(x) \neq 0$ for each $x \varepsilon I$. The Green's function for the ordin
differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$ is given by

$$G(x,t) = -\frac{1}{a_2(t)} \frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}$$

where y_1 and y_2 are linearly independent solutions of the correspond
homogeneous equation. Note that

$$\int_{x_0}^{x} G(x,t)\, f(t)\, dt$$

is a particular solution of the non-homogeneous equation. We have $m^2 + 5m + 6 = 0$ so that $y_1(x) = e^{-3x}$ and $y_2(x) = e^{-2x}$. Since

$$\begin{vmatrix} e^{-3x} & e^{-2x} \\ e^{-3t} & e^{-2t} \end{vmatrix} = e^{-(3x+2t)} - e^{-(3t+2x)}$$

and

$$\begin{vmatrix} e^{-3t} & e^{-2t} \\ -3e^{-3t} & -2e^{-2t} \end{vmatrix} = e^{-5t}.$$

Therefore,

$$G(x,t) = -e^{5t}(e^{-(3x+2t)} - e^{-(3t+2x)}) = e^{2(t-x)} - e^{3(t-x)}.$$

53. (A)

The Maclaurin series for e^u is given by

$$e^u = 1 + u + \frac{u^2}{2!} + \ldots + \frac{u^n}{n!} + \ldots$$

Setting $u = -x^2$ and multiplying the result by x yields the Maclaurin series for xe^{-x^2}:

$$xe^{-x^2} = x\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right)$$

$$= x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \ldots.$$

54. (E)

The symmetric difference is defined by

$$S\Delta T = (S\backslash T) \cup (T\backslash S).$$

We have

$$S\backslash T = \{1,2,3\}$$

and

$$T\backslash S = \{6,7,8\}$$

so that

$$S\Delta T = \{1,2,3,6,7,8\}.$$

55. (D)

Set $x_n = \alpha^n$ for $n = 0, 1, 2, 3, \ldots$. Then

$$\alpha^{n+2} + 6\alpha^{n+1} + 9\alpha^n = 0$$

$$\alpha^2 + 6\alpha + 9 = 0$$

$$\alpha = 3, 3.$$

Thus, the general solution is given by $x_n = a\,3^n + b\,n\,3^n$ where $a, b \in \mathbb{R}$. Since $x_0 = 1$ and $x_1 = 0$, we have

$$1 = x_0 = a$$

$$0 = x_1 = 3a + 3b$$

so that $a = 1$ and $b = -1$. Therefore, $x_5 = 3^5 - 5(3^5) = -4(3^5) = -972$.

56. (B)

If A represents the angle between f and g, then

$$\cos A = \frac{(f,g)}{\|f\|\,\|g\|}.$$

We have

$$\|f\|^2 = (f,f) = \int_0^1 2^2\, dx = 4 \qquad (\|f\| = 2)$$

$$\|g\|^2 = (g,g) = \int_0^1 x^2\, dx = \frac{1}{3} \qquad (\|g\| = 1/\sqrt{3})$$

$$(f,g) = \int_0^1 2x \ \ dx = 1 .$$

Therefore $\cos A = \sqrt{3}/2$.

57. (D)

The integral is improper since the integrand has a vertical asymptote at $x = 3$. Hence

$$\int_{-24}^{4} \frac{dx}{\sqrt[3]{(x-3)^2}} = \int_{-24}^{3} \frac{dx}{\sqrt[3]{(x-3)^2}} + \int_{3}^{4} \frac{dx}{\sqrt[3]{(x-3)^2}}$$

$$= \lim_{\delta \to 0^+} \int_{-24}^{3-\delta} \frac{dx}{\sqrt[3]{(x-3)^2}} + \lim_{\varepsilon \to 0^+} \int_{3+\varepsilon}^{4} \frac{dx}{\sqrt[3]{(x-3)^2}}$$

$$= \lim_{\delta \to 0^+} \left[3\sqrt[3]{x-3}\,\right]_{-24}^{3-\delta} + \lim_{\varepsilon \to 0^+} \left[3\sqrt[3]{x-3}\,\right]_{3+\varepsilon}^{4}$$

$$= 3 \lim_{\delta \to 0^+} \left[\sqrt[3]{-\delta} - \sqrt[3]{-27}\,\right] + 3 \lim_{\varepsilon \to 0^+} \left[\sqrt[3]{1} - \sqrt[3]{\varepsilon}\,\right]$$

$$= 12.$$

58. (A)

The statement is actually true for all positive integers. For $n = 1$, we have $1^2 + 2^2 + 3^2 = 36$ which is divisible by 9. Assume that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for some positive integer n. We have

$$(n+1)^3 + (n+2)^3 + (n+3)^3$$

$$= (n+1)^3 + (n+2)^3 + n^3 + 9n^2 + 27n + 27$$

$$= \{n^3 + (n+1)^3 + (n+2)^3\} + 9\{n^2 + 3n + 3\}$$

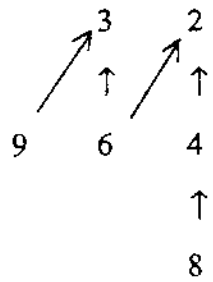
which is divisible by 9. Thus the statement is true for all positive integers.

59. (C)

If F is a finite field, then it must have p^n elements for p a prim n a non-negative integer. Since $27 = 3^3$, there exists a field wi elements.

60. (D)

The following diagram represents S with the given ordering



Therefore, the minimal elements of S are 6, 8, 9.

61. (B)

A subset of a topological space is a neighborhood of a poir contains an open set containing the point. Since $0 \varepsilon (-1, 1) \subseteq [$- and $(-1, 1)$ is open with respect to the usual topology on $\boldsymbol{R}$, [−1 a neighborhood of 0.

62. (E)

Let $\sigma \varepsilon S_n$, the symmetric group of degree n. If

$$\sigma = \sigma_m \ \sigma_{m-1} \ldots \sigma_i \ldots \sigma_2 \sigma_1$$

is the decomposition of σ into disjoint cycles where σ_i is a K_i- the Cauchy number of σ is given by

$$C(\sigma) = \sum_{i=1}^{m} (k_i - 1) .$$

We have $\sigma = (1\ 3\ 5\ 4)(2\ 6\ 7)$ Thus $C(\sigma) = (4-1)+(3-1) = 5$.

63. (B)

Consider the ordinary differential equation $M(x, y)\,dx + N(x,y)\,dy = 0$. If $M(x, y)$, $N(x, y)$, $\frac{\partial M(x,y)}{\partial y}$, and $\frac{\partial N(x,y)}{\partial x}$ are continuous in a region R, then the differential equation is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. For

$$(ye^{xy} + \cos x)\ dx + (xe^{xy} + 1)\ dy = 0$$

we have

$$\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy} = \frac{\partial N}{\partial x}.$$

64. (C)

Let x be a vertex of a graph G. The valence of x, denoted val(x), is the number of edges of G with x as one end point. The valences of the vertices and the number of vertices are related by $\text{val}(x_1) + \ldots + \text{val}(x_5) = 2e$ where e represents the number of edges. Thus $2 + 2 + 3 + 3 + 4 = 2e$ so that $e = 7$.

65. (D)

Since the discrete topology contains all subsets of $\mathbb{R}$, every subset of $\mathbb{R}$ is both open and closed. Therefore, the closure of (a, b) is $[a, b]$.

66. (A)

For $c, \hat{c} \in \mathbb{C}^3$ the usual inner product is given by

$$(c,\hat{c}) = x\,\overline{\hat{x}} + y\,\overline{\hat{y}} + z\,\overline{\hat{z}}$$

where $c = (x, y, z)$ and $\hat{c} = (\hat{x}, \hat{y}, \hat{z})$. Therefore $\hat{c}$ can be chosen that $\overline{\hat{x}} = 1$, $\overline{\hat{y}} = -i$, and $\overline{\hat{z}} = 2 + i$, that is $\hat{c} = (1, i, 2-i)$.