GRE MATHEMATICS
TEST III

TIME: 2 hours and 50 minutes
66 Questions

DIRECTIONS: Choose the best answer for each question and mark the letter of your selection on the corresponding answer sheet.

1. The generating function $f(x)$ for the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, ... is given by

(A) $(1-x+x^2-x^3)^{1/2}$

(B) $(1-x-x^2)^{-1}$

(C) $(1-x-x^2-x^3)^{-1}$

(D) $(1+x+x^2)^{-1/2}$

(E) $(1-x^2+x^3)^{-1}$

2. The number of solutions of $p(x)=x^2+3x+2$ in Z_6 is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

3. Which of the following matrices is normal? ($i = \sqrt{-1}$)

(A) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} i & 1 \\ -1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & i \\ -1 & 1 \end{bmatrix}$ (E) $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

4. Find the locus of all points (x, y), such that the sum of those distances from $(0, 1)$ and $(1, 0)$ is 2.

(A) $x^2 + xy + y^2 - 2x - 2y + 2 = 0$

(B) $3x^2 - 2xy + 3y^2 - 4x + 4y - 2 = 0$

(C) $4x^2 - 2xy + 4y^2 - 2x - 2y = 0$

(D) $3x^2 + 2xy + 3y^2 - 4x - 4y = 0$

(E) $x^2 - 2xy + y^2 + 2x - 2y + 4 = 0$

5. Find $\prod_{k=2}^{+\infty} \left(1 - \frac{1}{k^2}\right)$

(A) $\frac{1}{2}$ (D) $\frac{3}{4}$

(B) $\frac{1}{4}$ (E) $\frac{1}{8}$

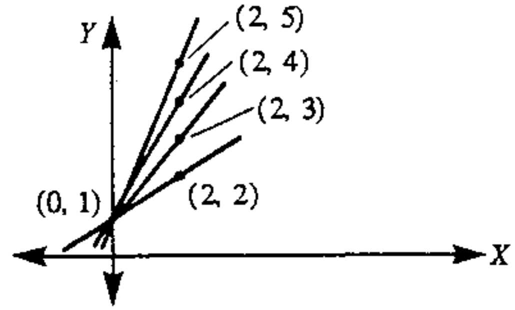
(C) 0

6. The cross ratio of the following set of lines is

(A) $\frac{4}{3}$ (D) $-\frac{5}{6}$

(B) $\frac{3}{2}$ (E) 1

(C) $\frac{1}{5}$



7. Find the characteristic of the ring $Z_2 + Z_3$.

(A) 0 (D) 4

(B) 6 (E) 2

(C) 3

8. Find $\lim_{n \to +\infty} \left(\sqrt{n^4 + i\, n^2} - n^2\right)$

(A) $\frac{i}{2}$ (D) $-\frac{1}{2}$

(B) 0 (E) $\sqrt{i}$

(C) $+\infty$

Which of the following is a solution of

$$u(x) = x + \int_0^x (t - x)\ u(t)\ dt\ ?$$

(A) $\sin x$

(B) $x \cos x$

(C) $\ln(x+1)$

(D) $x e^{-x}$

(E) $x e^{x}$

Find $\int_0^1 \left(\ln \frac{1}{x}\right)^5 dx$

(A) 120

(B) $+\infty$

(C) 1

(D) 720

(E) 24

Find the Laplace transform of

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1) \\ 1 & \text{if } x \in (1, +\infty) \end{cases}$$

(A) e^{-p}

(B) $\frac{1}{p}$

(C) $\frac{1}{e^{p-1}}$

(D) $p\,e^{p}$

(E) $\frac{1}{p\,e^{p}}$

12. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T(x, y) = \begin{bmatrix} 2x - y \\ x + 3y \end{bmatrix}$$

Find the adjoint T^* of T.

(A) $\begin{bmatrix} 2x + y \\ -x + 3y \end{bmatrix}$

(B) $\begin{bmatrix} x + 2y \\ x - 3y \end{bmatrix}$

(C) $\begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix}$

(D) $\begin{bmatrix} \frac{x}{2} - y \\ -x + \frac{y}{3} \end{bmatrix}$

(E) $\begin{bmatrix} 3x - y \\ x + 2y \end{bmatrix}$

13. The value of $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\, dx$ is

(A) 1

(B) $\frac{\pi}{2}$

(C) 0

(D) $\frac{\pi}{4}$

(E) π

14. Find the discriminant of the ternary quadratic form $x^2 - y^2 + z^2 - 2xy + 4yz - 6xz$

(A) 25

(B) 13

(C) 0

(D) 15

(E) 19

15. The radius of curvature of $f(x) = x + \frac{1}{x}$ at $P(1,2)$ is

(A) 1

(B) $\sqrt{2}$

(C) 4

(D) 2

(E) $\frac{1}{2}$

16. If $\Gamma(p)$ represents the gamma function, then $\int_0^{+\infty} e^{-x^2}\,dx$ is equal to $\frac{1}{2}\,\Gamma(p)$ when p is equal to

(A) -1

(B) $\frac{1}{2}$

(C) 1

(D) $\frac{1}{4}$

(E) 2

17. The factor group $\dfrac{(Z_2 \times Z_3)}{<(1,0)>}$ has order

(A) 2

(B) 3

(C) 4

(D) 1

(E) 6

18. Let $R\,[0, 1]$ denote the set of Riemann integrable func[tions] defined on [0, 1]. Which of the following is <u>not</u> satisfied by [the] function d defined on $R\,[0, 1]$ by

$$d(f,g) = \int_0^1 |f(x) - g(x)|\,dx\ ?$$

(A) $d(f, f) = 0$

(B) $d(f, g) \geq 0$

(C) $d(f, g) > 0$ if $f \neq g$

(D) $d(f, g) = d(g, f)$

(E) $d(f, g) \leq d(f, h) + d(h, g)$

19. Find the simple continued fraction for $\frac{13}{42}$.

(A) $\dfrac{1}{2 + \dfrac{1}{3 + \frac{1}{3}}}$

(B) $\dfrac{1}{3 + \dfrac{1}{3 + \frac{1}{4}}}$

(C) $\dfrac{1}{3 + \dfrac{1}{2 + \frac{1}{4}}}$

(D) $\dfrac{1}{4 + \dfrac{1}{4 + \frac{1}{3}}}$

(E) $\dfrac{1}{3 + \dfrac{1}{4 + \frac{1}{3}}}$

20. Which of the following polynomials satisfies an Eisenstein criterion for irreducibility over the rationals?

(A) $x^5 + 3x^4 + 18x^2 + 15x + 9$

(B) $2x^5 + 9x^4 + 15x^2 + 3x + 18$

(C) $3x^5 + 18x^4 + 15x^2 + 9x + 3$

(D) $4x^5 + 15x^4 + 9x^2 + 3x + 18$

(E) $5x^5 + 9x^4 + 18x^2 + 3x + 15$

21. The number of degrees that the conic, defined by

$$x^2 - y^2 + 2\sqrt{3}\, xy = 2$$

must be rotated in order to eliminate the xy term is

(A) 15 (D) 60

(B) 30 (E) 75

(C) 45

22. For the initial value problem $y'' + 6y' + 9y = 0$; $y(0) = 3$; $y'(0) = -11$ find £(y), the Laplace transform of y.

(A) $\dfrac{1}{p+3} + \dfrac{2}{(p+3)^2}$

(B) $\dfrac{-2}{p+3} - \dfrac{3}{(p+3)^2}$

(C) $\dfrac{3}{(p+3)^2}$

(D) $\dfrac{3}{p+3} - \dfrac{2}{(p+3)^2}$

(E) $\dfrac{1}{p+3} - \dfrac{2}{(p+3)^2}$

23. Find the join of the subgroups <4> and <6> of Z_{12}.

(A) <0> (D) <4>

(B) <2> (E) <5>

(C) <3>

24. For which n is the regular n-gon <u>not</u> constructible with a straightedge and compass?

(A) 6 (D) 17

(B) 9 (E) 20

(C) 15

25. Given $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, the sum of the elements in T^n is

(A) $3n$ (B) $n + 3$

(C) n (D) $2n$

(E) $n+2$

26. Let b: $\mathbb{R} \times \mathbb{R} \rightarrow$ be the bilinear form defined by

$$b(X;Y) = x_1y_1 - 2x_1y_2 + x_2y_1 + 3x_2y_2$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. Find the 2×2 matrix B of b relative to the basis $U = \{u_1, u_2\}$ where $u_1 = (0, 1)$ and $u_2 = (1,1)$.

(A) $\begin{bmatrix} 5 & -3 \\ 0 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ (E) $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$

27. Let $\mathcal{X} = \{a, b, c\}$. Which of the following classes of subsets of $\mathcal{X}$ does <u>not</u> form a topology on $\mathcal{X}$?

(A) $\{\mathcal{X}, \varnothing\}$

(B) $\{\mathcal{X}, \varnothing, \{a\}\}$

(C) $\{\mathcal{X}, \varnothing, \{a\}, \{b\}, \{a,b\}\}$

(D) $\{\mathcal{X}, \varnothing, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

(E) $\mathcal{P}(\mathcal{X})$, the power set of X

28. The eigenvalues for the initial value–eigenvalue problem

$$y'' + \lambda y = 0$$
$$y(0) = 0; \; y(\pi) = 0$$

are given by

(A) 1, 2, 3, 4, ... (D) 0, ±1, ±4, ±9, ±16, ...

(B) 1, 4, 9, 16, ... (E) ..., −3, −2, −1

(C) 0, ±1, ±2, ±3, ±4, ...

29. For switching functions f, g, and h, the expression $(f \vee g)(\bar{f} \vee h)$ is equivalent to

(A) $g\bar{f} \wedge gh$ (D) $g\bar{f} \vee fh \vee gh$

(B) $g \vee h$ (E) $g\bar{f} \wedge fh \vee gh$

(C) $g \wedge fh$

30. For the inner product $\langle A, B\rangle = \text{trace}(B'A)$ defined on the vector space of 2 by 2 matrices on $\mathbb{R}$, find the square of the norm of

$$T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

(A) 5 (B) 10

(C) 15

(D) 20

(E) 25

31. Find the limit of the series

$$x^4 + \frac{x^4}{1+x^2} + \frac{x^4}{(1+x^2)^2} + \frac{x^4}{(1+x^2)^3} + \dots$$

(A) $x^6 + x^4$

(D) $x^4 + \frac{x^6}{1+x^2}$

(B) $\frac{x^6}{1+x^2}$

(E) $x^4 + x^2$

(C) x^6

32. The domain of $f(x) = \int (x + 2x^2 + 3x^3 + \dots)\, dx$ is

(A) $(-1, 1)$

(D) $\left[\frac{1}{2}, \frac{1}{2}\right)$

(B) $[-1, 1)$

(E) $(-1, 1]$

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

33. Determine the number of homomorphisms from the group Z_8 onto the group Z_4.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

34. The solution of $x^2 y'' + 6xy' + 6y = 0$, for $x > 0$, is given by

(A) $c_1 x^3 - c_2 x^2$

(D) $c_1 e^{-2x} + c_2 e^{-3x}$

(B) $c_1 x + c_2 \ln x$

(E) $c_1 x \ln x + c_2 \ln x$

(C) $\frac{c_1}{x^3} + \frac{c_2}{x^2}$

35. The remainder of 5^{34} when divided by 17 is

(A) 0

(D) 6

(B) 2

(E) 8

(C) 4

36. Find the curl of $\overline{u} = xyz\,\vec{i} + xy^2\vec{j} + yz\,\vec{k}$

(A) $xz\,\vec{i} + (x - yz)\,\vec{j} + 2y\,\vec{k}$

(B) $(x - z)\,\vec{i} - yz\,\vec{j} + xyz\,\vec{k}$

(C) $z\,\vec{i} + xy\,\vec{j} + (y^2 - xz)\,\vec{k}$

(D) $xy\,\vec{i} - (z-y)\,\vec{j} + (xy - yz)\,\vec{k}$

(E) $(xy - yz)\,\vec{i} - yz\,\vec{j} + x\,\vec{k}$

37. The inverse of the function $f(x) = \dfrac{x}{x-1}$ is

(A) $\dfrac{x}{x+1}$ (D) $1 - \dfrac{1}{x}$

(B) $\dfrac{x}{x-1}$ (E) $\dfrac{x-1}{x+1}$

(C) $1 + \dfrac{1}{x}$

38. Find the slope of the tangent line to the ellipse

$$2x^2 + y^2 + 30 = 8y - 12x$$

at (x_0, y_0), where $x_0 = -2$ and $y_0 > 4$.

(A) $\dfrac{1}{\sqrt{2}}$ (D) $\dfrac{1}{2}$

(B) $-\sqrt{2}$ (E) -2

(C) 2

39. For matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix},$$

and $D = \begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix}$,

the matrix D is a linear combination $(aA + bB + cC)$ of A, B, C for a, b, c, given by

(A) $1, 1, -1$ (D) $1, -2, 1$

(B) $2, 1, -1$ (E) $-1, 1, -2$

(C) $2, 2, -2$

40. Given $p(x) = \sum_{k=1}^{+\infty} \dfrac{(x-2)^k}{k^2}$, find the interval in which $p'(x)$ converges.

(A) $\{2\}$ (D) $(1, 3]$

(B) $[1, 3)$ (E) $\mathbb{R} \to \mathbb{R}$

(C) $[1, 3]$

41. Find the Laplace transform of $\int_0^x \sin 2t \, dt$.

(A) $\dfrac{1}{p^2 + 4}$ (D) $\dfrac{4}{p^4 + 16}$

(B) $\dfrac{2p}{p^2 + 4}$ (E) $\dfrac{1}{p^2 + 2p}$

(C) $\dfrac{2}{p^3 + 4p}$

42. If $f'(x_0) = \sqrt{3}$, then the tangent line to the graph of f at x_0 makes an angle of β degrees with the positive x-axis. Find β.

(A) 0
(B) 30
(C) 45
(D) 60
(E) 90

43. Let $T: R^3 \to R^3$ be defined by $T(x, y, z) = (x + y,\ x - y + z,\ y + 2z)$. Find the trace of T.

(A) 5
(B) -1
(C) 0
(D) 7
(E) 2

44. Find the sum of $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots$.

(A) $\frac{3}{4}$
(B) 1
(C) $\frac{5}{4}$
(D) $\frac{3}{2}$
(E) $\frac{7}{4}$

45. Which of the following is not a proper ideal of the ring Z_{12}?

(A) <5>
(B) <8>
(C) <2>
(D) <3>
(E) <4>

46. Assuming that a person selects an answer to each of the first ten questions on this examination at random and that the selections are independent, what is the probability that he/she will guess exactly five answers correct?

(A) $\frac{(63)4^6}{5^{10}}$
(B) $\frac{(65)4^6}{5^{10}}$
(C) $\frac{4^9}{5^{10}}$
(D) $\frac{(61)4^6}{5^{10}}$
(E) $\frac{(67)4^6}{5^{10}}$

47. Find the Jacobian of the transformation from the xy-plane to the uv-plane defined by

$$u = f(x,y) = xe^{xy}$$
$$v = g(x,y) = ye^{xy}$$

(A) $2xye^{xy}$
(B) $(1 - x^2y^2)\, e^{2xy}$
(C) $2e^{2xy}$
(D) $(2xy + 1)\, e^{2xy}$
(E) 0

48. On average, a baseball player gets a hit in one out of three attempts. Assuming that the attempts are independent, what is the probability that he gets exactly three hits in six attempts?

(A) $\frac{160}{3^6}$

(B) $\frac{160}{3^5}$

(C) $\frac{1}{2}$

(D) $\frac{80}{3^6}$

(E) $\frac{40}{3^6}$

49. Find the number of units in the ring Z_5.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

50. Define $f(x) = x$ for $x \varepsilon (0, 1)$. Find the coefficient of the third term in the half range Fourier sine series.

(A) $\frac{2}{3\pi}$

(B) $\frac{4}{9\pi^2}$

(C) $\frac{2}{9\pi^2}$

(D) $\frac{1}{3\pi}$

(E) $\frac{1}{9\pi^2}$

51. Let V be the vector space of functions $f: R \to R$. Let S be the subspace generated by $\{e^x, e^{2x}, e^{-2x}\}$. Define D_x to be the derivative operator on S. Find the determinant of D_x.

(A) 2

(B) 0

(C) -4

(D) 1

(E) -1

52. Find Green's function for $y'' + 5y' + 6y = \sin x$

(A) $2e^{2(t-x)} + 3e^{3(t-x)}$

(B) $e^{2(t-x)} - e^{3(t-x)}$

(C) $e^{(t+x)} - e^{(t-x)}$

(D) $2e^{(t-x)} - 3e^{(t-x)}$

(E) $e^{3(x-t)} - e^{2(x-t)}$

53. The Maclaurin series for xe^{-x^2} is given by

(A) $x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \ldots$

(B) $x^3 - \frac{x^5}{2!} + \frac{x^7}{3!} - \frac{x^9}{4!}$

(C) $x - \frac{x^3}{2!} + \frac{x^5}{3!} - \frac{x^7}{5!} + \ldots$

(D) $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \ldots$

(E) $x + x^3 - \frac{x^5}{2!} + \frac{x^7}{3!} - \ldots$

The symmetric difference of the sets $S = \{1, 2, 3, 4, 5\}$ and $T = \{4, 5, 6, 7, 8\}$ is

(A) $\{4, 5\}$

(D) $\{3, 4, 5, 6\}$

(B) $\varnothing$

(E) $\{1, 2, 3, 6, 7, 8\}$

(C) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

Given

$$x_{n+2} + 6x_{n+1} + 9x_n = 0 \ (n = 0, 1, 2, \ldots)$$

$$x_0 = 1 \ ; \ x_1 = 0 \ ,$$

then $x_5 =$

(A) 576

(D) -972

(B) -834

(E) 774

(C) 1068

Let V be the vector space of real polynomials with inner product

$$(f, g) = \int_0^1 f(x) \ g(x) \ dx$$

where $f, g \ \varepsilon \ V$. Find the cosine of the angle between $f(x) = 2$ and $g(x) = x$.

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) 0

(D) 1

(E) $\frac{\sqrt{2}}{2}$

57. The integral $\int_{-24}^{4} \frac{dx}{\sqrt[3]{(x-3)^2}}$

(A) converges to 6

(D) converges to 12

(B) diverges to $+\infty$

(E) diverges to $-\infty$

(C) converges to 9

58. Let n be a positive integer greater than 3. Then $n^3 + (n+1)^3 + (n+2)^3$ is divisible by

(A) 9

(D) 6

(B) 4

(E) 15

(C) 12

59. If F is a finite field, then which of the following numbers can be the cardinality of F ?

(A) 21

(B) 45

(C) 27 (D) 14

(E) 33

(C) 3 (D) 4

(E) 5

60. Consider the set $S = \{2, 3, 4, 6, 8, 9\}$ ordered by "s is a multiple of t". How many minimal elements does S have?

(A) 0 (D) 3

(B) 1 (E) 4

(C) 2

61. Which of the following is a neighborhood of 0 relative to the usual topology τ for the real numbers?

(A) $(0, 1)$ (D) $[0, 1]$

(B) $[-1, 1]$ (E) $(-1, 0)$

(C) $[-1, 0]$

62. Find the Cauchy number for the permutation

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 1 & 4 & 7 & 2 \end{bmatrix} \varepsilon S_7.$$

(A) 1 (B) 2

63. Which of the following ordinary differential equations is exact?

(A) $(x + e^y)\, dx + (xe^y - 2xye^y - x^2 y)\, dy = 0$

(B) $(ye^{xy} + \cos x)\, dx + xe^{xy} + 1)\, dy = 0$

(C) $(\sin x \sin y + y^2)\, dx + (\cos x \cos y - 2xy)\, dy = 0$

(D) $(x^2 y - y^2)\, dx + (2 - xy)\, dy = 0$

(E) $(2x + 3y - 4)\, dx + (6x + 9y + 2)\, dy = 0$

64. Let G be a graph with vertices x_1, x_2, x_3, x_4, x_5. If $\mathrm{val}(x_1) = 2$, $\mathrm{val}(x_2) = 2$, $\mathrm{val}(x_3) = 3$, $\mathrm{val}(x_4) = 3$ and $\mathrm{val}(x_5) = 4$, where the valence of vertex x is denoted $\mathrm{val}(x)$, how many edges does G have?

(A) 4 (D) 8

(B) 9 (E) 5

(C) 7

65. If τ is the discrete topology on the real numbers R, find the closure of (a, b).

(A) (a, b)

(B) $(a, b]$

(C) $[a, b)$

(D) $[a, b]$

(E) R

66. Define $f: ¢^3 \to ¢$ by $f(c) = x - iy + (2 + i) z$ where $c = (x, y, z)$. Find a $\hat{c} \; \varepsilon \; c^3$ such that $f(c) = (c, \hat{c})$ for every $\hat{c} \; \varepsilon \; ¢^3$ where $(c, \hat{c})$ is the usual inner product on c^3.

(A) $(1, i, 2 - i)$

(B) $(-1, -i, 2 - i)$

(C) $(1, -i, 2 + i)$

(D) $(-1, i, -2 - i)$

(E) $(-1, i, -2 - i)$

GRE MATHEMATICS TEST III

ANSWER KEY

1.	B	23.	B	45.	A
2.	E	24.	B	46.	A
3.	D	25.	E	47.	D
4.	D	26.	D	48.	A
5.	A	27.	D	49.	E
6.	A	28.	B	50.	A
7.	B	29.	D	51.	C
8.	A	30.	C	52.	B
9	A	31.	E	53.	A
10.	A	32.	A	54.	E
11.	E	33.	C	55.	D
12.	A	34.	C	56.	B
13.	D	35.	E	57.	D
14.	D	36.	C	58.	A
15.	E	37.	B	59.	C
16.	B	38.	B	60.	D
17.	B	39.	B	61.	B
18.	C	40.	B	62.	E
19.	E	41.	C	63.	B
20.	E	42.	D	64.	C
21.	B	43.	E	65.	D
22.	D	44.	B	66.	A