# GRE MATHEMATICS TEST II

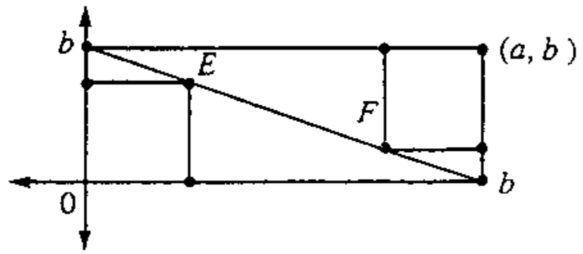
# DETAILED EXPLANATIONS OF ANSWERS

1. (B)

It is well-known that $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, so $(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$ by differentiation. Of course $|x| < 1$ is required for convergence. The given series is $\frac{1}{16}\sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^{n-1} = \frac{1}{16}\left(1-\frac{1}{4}\right)^{-2} = \frac{1}{9}$.

2. (A)

From the diagram shown, vertex $E$ must lie on the lines $y = x$ and $y = -\frac{b}{a}x + b$, so it has coordinates $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$, found by solving these equations simultaneously. By symmetry we must have coorindates for vertex $F$ as $\left(\frac{a^2}{a+b}, \frac{b^2}{a+b}\right)$.




Applying the distance formula,

$$d = \frac{\sqrt{a^2(a-b)^2 + b^2(b-a)^2}}{(a+b)}$$

which reduces to (A).

3. (A)

If $\lambda = (1\ 3\ 2)(4)(5)$, then $\lambda^2 = (1\ 2\ 3)(4)(5)$ and $\lambda^3 = (1)(2)(3)(4)(5)$, the identity. Hence the cyclic subgroup generated by $\lambda$ is of order 3. $A_5$ is of order $5!/2 = 60$ and so the required index is $60/3 = 20$.

4. (E)

The equation can be solved by separation of variables $\frac{dM}{M} = -\frac{dt}{10}$ and integration, yielding $\ln M = -\frac{t}{10} + C$. The value $M = M_0$ when $t = 0$ implies that $C = \ln M_0$. Taking the exponential of both sides of the solution yields

$$M = M_0 e^{-\frac{t}{10}} \quad \text{so } M(20) = M_0 e^{-2} = \frac{M_0}{e^2}.$$

5. (D)

First, we must have $f(1) = 1 - a + b = 0$. Then since $f'(1) = n - 1 + 2 - a = 0$ is also required, we must have $a = n + 1$ so $b = n$.

6. (B)

If $S$ is countable, then $S \cup T$ is the union of countable sets and hence countable. If $S$ is uncountable, then $S \cup T$ is uncountable as its cardinality is at least that of $S$. Counterexamples for the other cases:

(A) $S \cap T$ is countable but it may be finite.
For example, $S = \{0, 1, 2, \ldots\}$, $T = \{0, -1, -2, \ldots\}$.

(C) $\overline{S}$ may be uncountable and then $\overline{S} \cup T$ is also.

7. (E)

The amplitude of any wave in the form $a \sin kx + b \cos kx$ is $\sqrt{a^2+b^2}$. This follows since factoring $\sqrt{a^2+b^2}$ from the expression leaves coefficients $\frac{a}{\sqrt{a^2+b^2}}$ and $\frac{b}{\sqrt{a^2+b^2}}$ which have sum of squares equal to one. Hence we can rewrite the wave as

$$\sqrt{a^2+b^2}\ \sin(kx \pm \theta) \quad \text{where } \theta = \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right).$$

As $\sin(kx \pm \theta)$ can be at most 1, the maximum is at

$$\sqrt{a^2+b^2} = \sqrt{5^2+12^2} = \sqrt{169} = 13.$$

A calculus approach is much more difficult..

8. (C)

Each of the $P$ passes through the external loop creates $M + N$ operations in the internal loops. Basic counting technique implies the product $P(M+N)$, gives the total number of operations.

9. (B)

Directly, using the chain rule, $f'(x) = 3\,[g(x)]^2 \cdot g'(x)$ and then $f''(x) = 6\,g(x)[g'(x)]^2 + 3\,[g(x)]^2 g''(x)$. Now the coefficients of the series are related to $g(x)$ in general as $g^{(n)}(0) = n!a_n$. So $g(0) = a_0$, $g'(0) = a_1$ and $g''(0) = 2a_2$. Hence $f''(0) = 6a_0a_1^2 + 3a_0^2 \cdot 2a_2 = 6a_0(a_1^2 + a_0a_2)$.

10. (E)

$$f(f(x)) = f\left(\frac{x}{1-x}\right) = \frac{\frac{x}{1-x}}{\left(1-\frac{x}{1-x}\right)} = \frac{x}{1-x-x} = \frac{x}{1-2x}$$

Likewise,

$$f(f(f(x))) = f\left(\frac{x}{1-2x}\right) = \frac{\frac{x}{1-2x}}{\left(1-\frac{x}{1-2x}\right)} = \frac{x}{1-3x}$$

11. (A)

$S_1$ contains 1 and all unit fractions $1/n$. In all other $S_k$, only the numerators of the fractions change. Since the unit fraction $\frac{1}{n} = \frac{k}{nk}$ and this is in $S_k$, we find that

$$S_k \subseteq S_1 \quad \text{for all } k, \text{ so } \bigcap_{k=1}^{\infty} S_k = S_1.$$

12. (B)

It is simplest to view the limit as a defnite integral in the form

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x)\,\Delta x.$$

In this case, $\Delta x = 1/n$ and $a = 1$ so $f(x) = 1/x$. As usual, we can use $\Delta x(b-a)/n$ for interval $[a, b]$ to conclude that $b = 2$. Thus, the lmit is

$$\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2 = \ln 1 = \ln 2.$$

13. (E)

$$\int_0^a \frac{x^2+b^2}{x^2+a^2}\,dx = \int_0^a \left(\frac{x^2+a^2}{x^2+a^2} + \frac{b^2-a^2}{x^2+a^2}\right) dx$$

$$= \int_0^a \left(1 + \frac{b^2-a^2}{x^2+a^2}\right) dx$$

$$= \left(x + \frac{b^2-a^2}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)\Big|_0^a$$

$$= a + \frac{\left(b^2-a^2\right)}{a}\frac{\pi}{4}$$

$$= \frac{4a^2}{4a} + \frac{\left(b^2-a^2\right)\pi}{4a}$$

$$= \frac{4a^2 + b^2\pi - a^2\pi}{4a}$$

$$= \frac{(4-\pi)a^2 + b^2\pi}{4a}$$

$$= (4-\pi)\frac{a}{4} + \frac{\pi b^2}{4a}.$$

14. (E)

For a relative minimum, we must simultaneously have $f_x = 3x^2 - ay - 1 = 0$, $f_y = -ax + 2y = 0$, and $D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 12x - a^2 > 0$ since $f_{yy} = 2 > 0$ independent of $x, y$. The first two conditions imply $y = ax/2$ and $3x^2 - \frac{a^2}{2}x - 1 = 0$. Solving the quadratic always yields two roots,

$$x = \frac{\left(\frac{a^2}{2} \pm \sqrt{\frac{a^4}{4} + 12}\right)}{6}$$

where the + sign will yield the necessary positive root for $D > 0$. Then $D = a^2 + \sqrt{a^4+48} - a^2 = \sqrt{a^4+48} > 0$ independent of the choice of a. No lower bound exists.

15. (D)

The result (A) is a standard theorem in vector spaces. For (C), we need only note that subspaces are closed under scalar multiplication so $2S \subseteq S$. But, for every $x$ in $S$, $1/2\,x$ is in $S$ and $2(1/2\,x) = x$ so $S \subseteq 2S$, and we can conclude that $S = 2S$, so $2S$ is a subspace.

To show that (B) is not necessarily true, let the vector space be $R^2$, $S = \{ (a, 0) \mid a \in R \}$, $T = \{ (a, b) \mid a + b = 0,\ a, b \in R \}$. Then $S$ and $T$ are subspaces of $R^2$ (this is easily checked), and $S \cup T = \{ (a, b) \mid b = 0 \text{ or } a + b = 0, a, b \in R \}$. We see that $S \cup T$ is <u>not</u> a subspace of $R^2$ because it is not closed under addition. For example, if we take $s, t \in S \cup T$ where $s = (1, 0)$ and $t = (1, -1)$, then $s + t = (2, -1) \notin S \cup T$.

16. (E)

The Cauchy-Riemann conditions to be satisfied are

$$\frac{\partial(x^2+y^2)}{\partial x} = \frac{\partial(2x)}{\partial y} \quad \text{and} \quad \frac{\partial(x^2+y^2)}{\partial y} = -\frac{\partial(2x)}{\partial x},$$

which imply $x = 0$ and $y = -1$.

17. (D)

If $y = f(x)$ is differentiably redundant of order $n$ then $y^{(n)} - y = 0$ and solutions are related to the $n$th roots of unity. Clearly, the sum of two functions, one of order $m$ and the other of order $n$, yields a function of order $l\,\text{cm}(m, n)$. This eliminates (B) which must be of order 4. (A) is of order 6 since $-1/2 \pm i\sqrt{3}/2$ are 3rd roots of unity. (C) is of order

6 since $1/2 \pm i\sqrt{3}/2$ are 6th roots of unity. In each of these latter cases we have primitive roots and this is necessary. For instance $-1$ is also a 6th root of unity but $e^{-x}$ is not of order 6.

18. (E)

The dimension of the solution space of a matrix always is given by the number of columns (7) or unknowns less the rank of the matrix (4).

19. (D)

Non-isomorphic abelian groups of the same order, $n$, are effectively the direct products $Z_{n1} \times Z_{n2} \times \ldots \times Z_{nk}$ where $n_1 \times n_2 \times \ldots n_k = n$ and each $n_i$ is a divisor of $n$. In this case, the products yielding 40 are 40, $10 \times 4$, $8 \times 5$, $20 \times 2$, $10 \times 2 \times 2$, $5 \times 4 \times 2$, and $5 \times 2 \times 2 \times 2$.

20. (B)

It may simply be well known that half the sums will be even (including sum zero for the empty set) and the number of subsets is $2^9 = 512$, so $1/2(512) - 1 = 255$ (for the empty set). A more direct anaylsis follows in that an even sum results in each of the $2^4 = 16$ subsets containing only the even integers, 2, 4, 6, 8. These may be combined with either 0, 2, or 4 odd numbers and the number of ways to choose these is

$$\binom{5}{0} = 1, \binom{5}{2} = 10, \text{ or } \binom{5}{4} = 5, \text{ respectively.}$$

Hence the total $16(1 + 10 + 5) = 16^2 = 256$ which includes the empty set.

21. (B)

Given $\overline{X} = \{x, y\}$, every topology must contain the oper $\overline{X}$ and $\phi$, the empty set. When these are the only open sets, we one (trivial) topology. The other three possible topologies are { $\{\{y\}\}$, and $\{\{x\}, \{y\}\}$.

22. (B)

This is very tedious by direct means, but generally the boun $/|f'(c)|$ when $f$ is differentiable and $f'(c) \neq 0$, on an interval conta $c$. In this case $f'(2) = 23$.

23. (B)

Since $3^3 = 27 \equiv (-1) \pmod 7$ we find $3^{18} \equiv (-1)^6 \equiv 1 \pmod 7$, $\equiv 3^2 \pmod 7 \equiv 2 \pmod 7$.

24. (C)

The Heaviside method yields the coefficient

$$\left.\frac{s^2+1}{s^2-2}\right|_{s^2=-3} = \frac{-2}{-5} = \frac{2}{5}.$$

25. (D)

By substitution $y = a\,(ay^2 + 2)^2 + 2$. Then $y = 1$ implies $a(a$ $+ 1 = 0$ or $a^3 + 4a^2 + 4a + 1 = 0$ with obvious solution $a = -1$ yie the factor $a^2 + 3a + 1$ with two real (irrational) roots.

26. (D)

$H \cap K$ is always a subgroup and invariance follows easily. HK is a subgroup since invariance guarantees that $HK = KH$. Finally, $HK$ is invariant since $gHK = HgK = HKg$, for all $g \in G$.

27. (E)

Since $x = x^2$ we must have $x + y = (x + y)^2 = x^2 + y^2 + xy + yx = x + y + xy + yx$. So $xy + yx = 0$. When $x = y$ we get $2x^2 = 2x = 0$ so $x = -x$. Combining we find $xy = -yx = yx$ and $R$ is commutative.

28. (B)

For $a = r + s\sqrt{17} \in D$ let $\overline{a} = r - s\sqrt{17} \in D$. The norm of $a$ is $N(a) = a\overline{a} = r^2 - 17s^2$. It is well known that $N(a)N(b) = N(ab)$ and this implies that whenever $N(a)$ is prime then $a$ is irreducible. The norms of the choices show only one prime, $N(9 - 2\sqrt{17}) = 81 - 68 = 13$.

29. (E)

Direct substitution yields

$$k\,(k-1)\,x^k - 3kx^k - 4x^k = 0,$$

and since $x$ is not always 0, we can claim that $k^2 - 4k - 4 = 0$ and solve for $k = 2 \pm 2\sqrt{2}$.

30. (C)

Since $(x^2 + 1)^2 = x^4 + 2x^2 + 1$ we can find that

$$x^4 + 1 = (x^2 + 1 + \sqrt{2}\,x)(x^2 + 1 - \sqrt{2}\,x).$$

31. (C)

This slope is the directional derivative of $z$ at $(1, 3)$ in the direction given by $\hat{i} + 3\hat{j} = \overline{v}$ (a vector of slope 3). To compute this, we need the gradient vector $\nabla z = z_x\hat{i} + z_y\hat{j}$ or $2x\hat{i} - 2y\hat{j} = 2\hat{i} - 6\hat{j}$. Then the directional derivative is

$$\nabla z \cdot \frac{\overline{v}}{|\overline{v}|} = \frac{(2 - 18)}{\sqrt{10}} = -8\sqrt{\frac{2}{5}}.$$

32. (B)

Since the row sums are all equal to $a + (n-1)\,b$ this must be an eigenvalue corresponding to the eigenvector, $\overline{x} = (1, 1, \ldots, 1)$.

33. (E)

The correct relationship is

$$\frac{\text{VOLUME}}{\text{AREA}} = \frac{\frac{(4\pi r^3)}{3}}{\pi r^2} = \frac{4}{3}r.$$

The formula should be

$$\frac{4}{3}\,\frac{d}{2}\left(\frac{8}{9}d\right)^2 = \frac{2^7 d^3}{3^5}.$$

34. (A)

To have eigenvalue $\lambda = 1$ we must have $|A - I| = 0$.

$$\Rightarrow \begin{vmatrix} k-1 & 1 & 2 \\ 1 & 1 & k \\ 1 & 2 & 2 \end{vmatrix} = 2k^2 - 5k + 2 = (2k-1)(k-2) = 0,$$

so $k = 1/2$ or $k = 2$ will suffice.

35. (A)

In polar form, $z = e^{i\pi/4}$ and $z^{14} = e^{i\,14\pi/4}$. The periodic quality of $e^{x}$ means $z^{14} = e^{-i\pi/2} = -i$.

36. (B)

For a solution of relatively small magnitude, as proposed, we may approximate $e^{-x/100}$ by $e^0 = 1$. Then solving $1 - e^{-x} = e^{-1}$ implies $e^{-x} = 1 - e^{-1}$ or $-x = \ln(1 - e^{-1})$ or

$$x = \ln\left(\frac{e}{(e-1)}\right) = 1 - \ln(e-1) = 1 - \ln(1.718+\ ).$$

Since $\sqrt{3} = 1.732+$ the best choice is $1 - \ln\sqrt{3} = 1 - \frac{1}{2}\ln 3$.

37. (D)

The necessary algebra is

$$\frac{1}{z-4} = \frac{1}{z-1-3} = \frac{\frac{1}{(z-1)}}{1-\left[\frac{3}{(z-1)}\right]} = \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{3}{z-1}\right)^n,$$

by the formula for the sum of a geometric series. Then the coefficient corresponds to $n = 1$ in the form

$$\sum_{n=0}^{\infty} 3^n (z-1)^{-n-1}.$$

38. (A)

The Fundamental Theorem of Calculus states that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Then by the chain rule we must have

$$\frac{d}{dx}\left[\int_a^{x^2} f(t)\,dt\right] = 2x\,f(x^2) = \frac{2x}{(1+x^6}$$

in this case. Evaluation at $x = 2$ yields $\frac{4}{(1+64)}$ =

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The latter sum is geometric and its corresponding se

$$\frac{1}{18}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{12}.$$

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Since the entries in each row are one unit apart, th row equivalent to the matrix shown where the rank visible. Actually $n \geq 2$ could be used without changin

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41. (B)

The inverse $x$, must satisfy the equation $10x \equiv 1\ (17)$ in modular arithmetic or $10x = 17y + 1$. Apparently $17y$ ends in the digit 9, and so $y = 7$ must be the case. Then $10x = 119 + 1$ implies $x = 12$.

42. (E)

By substitution of $y = x + b$ from the latter equation we get $x^2 - x - b = a$ or $x^2 - x - (a + b) = 0$ in the former. The quadratic formula implies two distinct solutions for $x$ unless $(a + b) = -1/4$, and then $x = 1/2$ follows from $(x - 1/2)^2 = 0$.

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By basic properties, $\log_4 64 = \log_4 4^3 = 3$. Clearly, $\log_7 343 = \log_7 7^3 = 3$. Also a standard identity for change of base is $\log_b x = \dfrac{\log_a x}{\log_a b}$, so (B) is also correct.

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The points on the unit circle corresponding to arcs of integral measurement (whole radians) are dense in the circle. We know that $y$ is a limit point of $S$ if every neighborhood of $y$ contains $x \neq y$ such that $x \varepsilon S$. Since the whole radians are dense in the circle, the values of tan $k$, $K = 1, 2, 3, \ldots$ are dense in the real numbers. Hence if $y$ is any real number, then any neighborhood of $y$ will also contain a value of tan $(k) \neq y$, $k = 1, 2, 3, \ldots$, so the limit points of $S$ consist of all the points on the real line. This means that the entire range $(-\infty, \infty)$ of the real function $f(x) = \tan x$ is the set of limit points.

45. (E)

As in problem 44, the points on the unit circle corresponding to whole radians are dense in the circle. Hence the values of $\sin(n)$, $n = 1, 2, 3, \ldots$ are dense in the range of $\sin n$, namely $[-1, 1]$. Therefore, $\lim_{n \to \infty} \inf\{\sin n\} = \inf\{-1, 1\} = -1$ and $\lim_{n \to \infty} \sup\{\sin n\} \sup\{-1, 1\} = 1$ so the difference is $-1 -1 = -2$.

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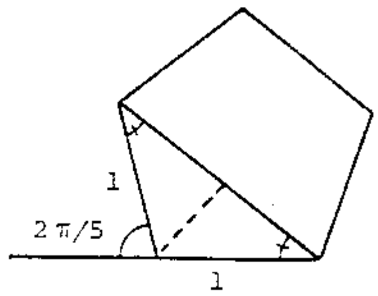
In the diagram, the diagonal is the base of an isosceles triangle with external angle $2\pi/5$. The base angles must be $\pi/5$. Therefore, the base is $2 \cos \pi/5$.




47. (D)

The characteristic polynomial of the equation is $m^2 - 2m - 8 = (m - 4)(m + 2)$, implying that the general form of all solutions is $y = c_1 e^{4x} + c_2 e^{-2x}$. Only (D) is in this form.

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If $f(x)$ is strictly increasing then $g(x) = -f(x)$ must be strictly decreasing and hence has no maximum since $g(a)$ is undefined.

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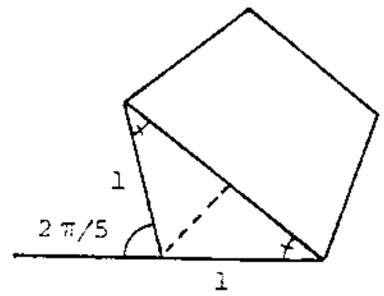


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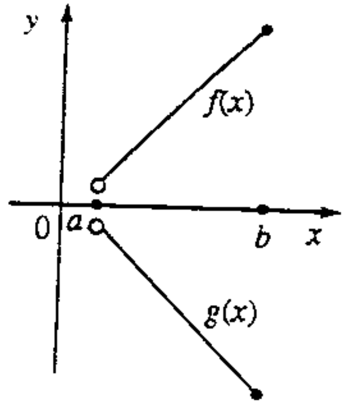



49. (B)

Permutations of $n = n_1 + n_2 + n_3$ objects, when groups of $n_1$, $n_2$, and $n_3$ are not distinguishable, total $\frac{n!}{n_1!n_2!n_3!}$. In this case $n = 6$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$. So we get $\frac{6!}{1!2!\,3!} = \frac{720}{12} = 60$.

50. (A)

There are two possible structures. In the first, the side of length 5 is the base, and then the median from the opposite vertex is 3 times the inradius in length so the area is $3(3/2)(5/2) = 45/4$. In the second, the side of length 5 is one of two equal sides, and then the base must be $2\sqrt{25 - 9(3/2)^2} = \sqrt{19} < 5$ so the maximum is 45/4.

51. (B)

In spherical coordinates, $\phi$ represents the angle from the positive $z$-axis to the line generating a cone with $z$-axis of symmetry and center at the origin. In spherical coordinates $\rho^2 = x^2 + y^2 + z^2 = 4$, so $\rho = 2$.

Also $2\cos\Phi = z = -\sqrt{x^2 + y^2}$

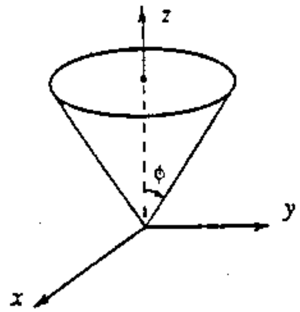
$$= -\sqrt{\rho^2\sin^2\Phi\cos^2\Phi + \rho^2\sin^2\Phi\sin^2\theta}$$

$$\Rightarrow 2\cos\Phi = -\sqrt{\rho^2\sin^2\Phi} = -\rho\,\sin\Phi = -2\sin\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = -1$$

$$\Rightarrow \theta = \frac{3\pi}{4}.$$

So in this case, the surfaces, in spherical coordinates, have equ $\rho = 2$, (a sphere) and $\Phi = 3\pi/4$ (a lower half-cone).




52. (B)

Each subcomponent has reliability $1-(1-R)^2 = R(2-R)$. Th series 0 | $n$ subcomponents must have reliability $R^n(2-R)^n$.

53. (D)

For the given family, $y' = y/x$; thus, the orthogonal family s the differential equation $y' = -x/y$. In separated form, we ha $+ y\,dy = 0$, which can be integrated directly to yield $\frac{x^2}{2} + \frac{y^2}{2}$ where $C$ is arbitrary, so $x^2 + y^2 = 2C = r^2$, giving circles cent (0, 0).

54. (E)

In standard form the linear difference equation is $x_{n+2} - x_{n+1} - 6x_n = -36n$, which implies characteristic roots 3 and $-2$ and a particular solution in the form $An + B$. Substitution leads to matching of coefficients via $-6An + A - 6B = -36n$, so $-6A = -36$ and $A = 6$. Then $6 - 6B = 0$ and $B = 1$. All solutions must be in the form $x_n = c_1 3^n + c_2 (-2)^n + 6n + 1$ where $c_1$, $c_2$ are arbitrary constants. (E) is in this form with $c_1 = 0$ and $c_2 = 2$.

55. (A)

Continuity implies $f^{-1}(U)$ must be open in $\overline{X}$ for every open set $U$ in $\overline{X}$. But then $f(f^{-1}(U)) = U$ shows $U$ is the image under $f$ of an open set. Since $f$ is a surjection, $f^{-1}(U)$ is well-defined for every open set. A counterexample for both (B) and (C) is $f(x) = e^x \sin x$ on the reals.

56. (D)

The negative of the gradient vector always supplies this direction. Here

$$-\nabla z = -y^2\hat{i} + 2xy\hat{j})\Big|_{(2,-1)}$$
$$= -\hat{i} + 4\hat{j}.$$

57. (D)

Euler's formula states that $V - E + F = 2$. Here $V = 27$ and $E = 40$ so $F = 15$.

58. (C)

The term in the expansion is given combinatorially by

$$\binom{5}{3} x^2(-2y)^3 = -80\, x^2y^3 .$$

59. (B)

The partial fractions expansion multiplied by $x$ gives

$$x\left[\frac{A}{x} + \frac{B}{2+3x} + \frac{C}{(2+3x)^2}\right] = \frac{x}{x(2+3x)^2}$$

Taking limits as $x \to +\infty$ yields $A + B/3 = 0$. $A = 1/4$ by the following calculations:

$$\frac{A}{x} + \frac{B}{2+3x} + \frac{Cx+D}{(2+3x)^2} = \frac{1}{x(2+3x)^2}$$

$$\Rightarrow A(2+3x)^2 + Bx(2+3x) + (Cx+D)x = 1$$

$\Rightarrow 4A = 1$, since all other terms will contain a factor of $x$,

$$\Rightarrow A = \frac{1}{4}.$$

So $B = -3/4$ and the integral of this term is $\left(-\frac{3}{4}\right)\left(\frac{1}{3}\right) \ln |2 + 3x|$.

60. (D)

It is well known that in the waiting time model the probability of a tail will be $1/T$. Then the probability of at least two tails is the complement of zero or one tail. As usual, this is

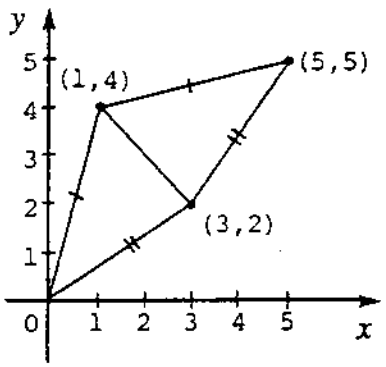
$$1 - \left(1 - \frac{1}{T}\right)^{3T} - 3T\left(1 - \frac{1}{T}\right)^{3T-1}\left(\frac{1}{T}\right).$$

Using $T^{3T}$ as a common denominator yields the numerator

$$T^{3T} - (T-1)^{3T} - 3T(T-1)^{3T-1}$$
$$= T^{3T} - (T-1)^{3T-1}(T - 1 + 3T).$$

61. (B)

The quadrilateral can be split into two triangles by the diagonal joining $(1, 4)$ and $(3, 2)$ as shown. The triangles have simple centroidal first coordinates 4/3 and 9/3 as the average of those for their vertices. The sides of the triangles are identical, so the average of these is the centroid coordinate $4/6 + 9/6 = 13/6$.




62. (A)

All that is needed is to show closure under the operations. For (A), if $xr = 0$ and $xs = 0$ then $xr + xs = 0 + 0 = 0$ and $(xr)(xs) = 0 \cdot 0 = 0$.

(B) will not contain 0 and this is a must for a subring.

(C) may not contain 0 if $x$ is not nilpotent.

(D) fails to guarantee that $(mx)(nx) = mnx^2$ is a multiple of $x$. For example, let $R$ be the ring of $2 \times 2$ matrices.

63. (E)

The limit can be simplified as

$$\lim_{m \to 0} \frac{1}{m}\left[\lim_{n \to 0} \frac{f(2+m+n) - f(2+m)}{n} - \lim_{n \to 0} \frac{f(2+n) - f(2)}{n}\right]$$
$$= \lim_{m \to 0}\left[\frac{f'(2+m) - f'(2)}{m}\right] = f''(2) = \frac{-1}{2^2} = -\frac{1}{4}$$

by the definition of derivative since $f$ is twice differentiable at $x = 2$.

64. (A)

Every open set containing a cluster point of $E$ must contain some <u>other</u> point in $E$. Since $\{x_1\}$ is open, $x_1$ can never be a cluster point. Likewise $\{x_1, x_2\}$ is open and contains only $x_2$ in $E$ so $x_2$ is not a cluster point. For $x_n$, $n \geq 3$, every open set containing $x_n$ also contains $x_2$ in $E$, so these are all cluster points.

65. (C)

The weights can be thought to act as their centers. Further, the proportion of the weight supported at the left is the ratio of the distance to the right support and the total distance between supports. In this case, we get $100\,(16/20) + 200(3/20) = 80 + 30 = 110$.

66. (A)

The series may diverge if $a_n$ is not a series of terms all of the same sign. For example, let $a_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum_{n=1}^{\infty} \cdot a_n$ converges by the alternating series test. However, $\sum_{n=1}^{\infty} a_n^2$ is the well-known divergent harmonic series.