# GRE MATHEMATICS TEST II

**TIME:** 2 hours and 50 minutes
66 Questions

**DIRECTIONS:** Choose the best answer for each question and mark the letter of your selection on the corresponding answer sheet.

1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{4^{n+1}}$.

(A) $\frac{1}{12}$

(B) $\frac{1}{9}$

(C) $\frac{1}{6}$

(D) $\frac{4}{3}$

(E) $\frac{1}{3}$

2. A rectangle has dimensions $a$ units by $b$ units with $a > b$. A diagonal divides the rectangle into two triangles. A square, with sides parallel to those of the rectangle, is inscribed in each triangle. Find the distance between the vertices (of the squares) that lie in the interior of the rectangle.

(A) $\frac{(a-b)\sqrt{a^2+b^2}}{a+b}$

(B) $\sqrt{a^2-b^2}$

(C) $\frac{a^2-b^2}{\sqrt{ab}}$

(D) $\frac{a^2-b^2}{\sqrt{a^2+b^2}}$

(E) $\frac{(a-b)\sqrt{a^2-b^2}}{a+b}$

3. Find the index of the subgroup generated by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

in the alternating group $A_5$.

(A) 20 (D) 3

(B) 40 (E) 5

(C) 24

4. In a pure radioactive decay model, the rate of change in the mass, $M$, satisfies the differential equation, $dM/dt = -M/10$. If the initial value of $M$ is $M_0$, find $M$ in terms of $M_0$ after 20 units of time, $t$, have elapsed.

(A) $\frac{1}{2} M_0$ (D) $\frac{M_0}{e}$

(B) $\frac{1}{4} M_0$ (E) $\frac{M_0}{e^2}$

(C) $\frac{M_0}{2e}$

5. Let $f(x) = x^{2n} - x^{2n-1} + \ldots + x^4 - x^3 + x^2 - ax + b$. Which value for the pair $(a, b)$ will insure that the $x$-axis will be tangent to the graph of $f(x)$ at $x = 1$?

(A) $(1, 1)$ (B) $(n, n-1)$

(C) $(n-1, n)$ (D) $(n+1, n)$

(E) $(n, n+1)$

6. Let the set $S$ be infinite and let the set $T$ be countably infinite. Let $\overline{S}$ denote the complement of $S$. If $S$ and $T$ are both subsets of the real numbers, which of the following pairs of sets must be of the same cardinality?

(A) $T$ and $S \cap T$ (D) Both (A) and (B)

(B) $S$ and $S \cup T$ (E) Both (A) and (C)

(C) $T$ and $\overline{S} \cup T$

7. Find the maximum value of $f(x) = 5 \sin 7x + 12 \cos 7x$.

(A) 12 (D) 17

(B) 5 (E) 13

(C) 7

8. In a computer program, separate loops with distinct indices produce $M$ and $N$ operations respectively. If these reside internally in a loop with an independent index producing $P$

operations, find the total number of operations represented by the three loops.

(A) $P^{M+N}$

(B) $(M+N)^P$

(C) $P(M+N)$

(D) $P^M + P^N$

(E) $M^P + N^P$

9. Give that the Taylor series for $g(x)$ is $\sum_{n=0}^{\infty} a_n x^n$ and $f(x) = [g(x)]^3$, find $f''(0)$.

(A) $3a_0(2a_1^2 + a_0 a_2)$

(B) $6a_0(a_1^2 + a_0 a_2)$

(C) $3a_0(2a_1 + a_0 a_2)$

(D) $6a_0(a_1 + a_0 a_1)$

(E) None of these

10. If $f(x) = x/(1-x)$, find $f(f(f(x)))$.

(A) $\dfrac{x^3}{1-x^3}$

(B) $\dfrac{3x}{3-x}$

(C) $\dfrac{x}{3-x}$

(D) $\dfrac{3x}{1-3x}$

(E) $\dfrac{x}{1-3x}$

11. Define $S_k = \{\frac{k}{j} \mid j = k, k+1, \ldots\}$ for $k = 1, 2, \ldots$. Find th intersection $\bigcap_{k=1}^{\infty} S_k$.

(A) $S_1$

(B) the empty set

(C) the interval $[0, 1]$

(D) the interval $(0, 1)$

(E) $\{\frac{k}{j} \mid 0 < \frac{k}{j} \le 1\}$

12. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{1}{n(1+\frac{k}{n})}$.

(A) 1

(B) $\ln 2$

(C) 0

(D) $+\infty$

(E) Does not exist

13. Evaluate the definite integral $\int_0^a \dfrac{x^2+b^2}{x^2+a^2}\,dx$.

(A) $\dfrac{((4-\pi)a + \pi b^2)}{4a}$

(B) $4a^2 + \pi(b^2 - a^2)$

(C) $\dfrac{((2-\pi)a + \pi b^2)}{2a}$

(D) $a + (b^2 - a^2)\ln 2$

(E) $(4-\pi)\dfrac{a}{4} + \dfrac{\pi b^2}{4a}$

14. Let $f(x, y) = x^3 - axy + y^2 - x$. Find the greatest lower bound for $a$ so that $f(x, y)$ has a relative minimum point.

(A) 0
(B) $\sqrt{48}$
(C) 12
(D) 6
(E) Does not exist

15. Given that $S$ and $T$ are subspaces of a vector space, which of the following is also a subspace?

(A) $S \cap T$
(B) $S \cup T$
(C) $2S$
(D) Both (A) and (C)
(E) Both (B) and (C)

16. Find the point $x + iy$ at which the complex function $f(x + iy) = x^2 + y^2 + 2xi$ is differentiable.

(A) 0
(B) $i$
(C) $1 - i$
(D) $1 + i$
(E) $-i$

17. Define a function, $f(x)$, to be <u>differentiably redundant</u> of order $n$ if the $n$-th derivative $f^{(n)}(x) = f(x)$ but $f^{(k)}(x) \neq f(x)$ when $k < n$. For easy examples, in this context, $e^x$ is of order 1, $e^{-x}$ is of order 2, and $\cos x$ is of order 4. Which of the following functions is differentiably redundant of order 6?

(A) $e^{-x} + e^{\frac{-x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)$

(B) $e^{-x} + \cos x$

(C) $e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)$

(D) Both (A) and (C)

(E) (A), (B) and (C)

18. In a homogeneous system of 5 linear equations in 7 unknowns, the rank of the coefficient matrix is 4. The maximum number of independent solution vectors is

(A) 5
(B) 2
(C) 4
(D) 1
(E) 3

19. The number, up to isomorphism, of abelian groups of order 40 is

(A) 40
(B) 20

(C) 8

(D) 7

(E) 5

20. From the set of integers $\{1, 2, \ldots, 9\}$, how many nonempty subsets sum to an even integer?

(A) 512

(D) $\frac{9!}{2!}$

(B) 255

(E) None of these

(C) $\frac{9!}{2!\,7!}$

21. How many topologies are possible on a set of 2 points?

(A) 5

(D) 2

(B) 4

(E) 1

(C) 3

22. In the $(\varepsilon, \delta)$ definition of the limit, $\lim_{x \to c} f(x) = L$, let $f(x) = x^3 + 3x^2 - x + 1$ and let $c = 2$. Find the least upper bound on $\delta$ so that $f(x)$ is bounded within $\varepsilon$ of $L$ for all sufficiently small $\varepsilon > 0$.

(A) $\frac{\varepsilon}{8}$

(D) $\frac{(\varepsilon^2)}{16}$

(B) $\frac{\varepsilon}{23}$

(E) $\frac{\varepsilon}{19}$

(C) $\frac{(\varepsilon^3)}{4}$

23. Find the remainder on dividing $3^{20}$ by 7.

(A) 1

(D) 4

(B) 2

(E) 5

(C) 3

24. In the partial fractions expansion of $\frac{s^2 + 1}{(s^2 - 2)(s^2 + 3)}$, the numerator of the fraction with denominator $s^2 + 3$ is

(A) 3

(D) $2s + 1$

(B) $\frac{3}{5}$

(E) $1 - 2s$

(C) $\frac{2}{5}$

25. For how many distinct real coefficients, $a$, will the system of equations $y = ax^2 + 2$ and $x = ay^2 + 2$ admit a solution with $y = 1$ ?

(A) 0

(D) 3

(B) 1

(E) 4

(C) 2

26. A subgroup $H$ in group $G$ is invariant if $gH = Hg$ for every $g$ in $G$. If $H$ and $K$ are both invariant subgroups of $G$, which of the following is also an invariant subgroup?

(A) $H \cap K$
(B) $HK$
(C) $H \cup K$
(D) Both (A) and (B)
(E) Both (B) and (C)

27. Let $R$ be a ring such that $x^2 = x$ for each $x \in R$. Which of the following must be true?

(A) $x = -x$ for all $x \in R$
(B) $R$ is commutative
(C) $xy + yx = 0$ for all $x, y \in R$
(D) Both (A) and (C)
(E) (A), (B), and (C)

28. In the integral domain $D = \{r + s\sqrt{17} \mid r, s \text{ integers}\}$, which of the following is irreducible?

(A) $8 + 2\sqrt{17}$
(B) $3 - \sqrt{17}$
(C) $9 - 2\sqrt{17}$
(D) $7 + \sqrt{17}$
(E) $13 + \sqrt{17}$

29. For which value of $k$ is $x^k$ a solution for the differential equation $x^2y'' - 3xy' - 4y = 0$ ?

(A) 4
(B) 3
(C) 2
(D) 1
(E) None of these

30. Which of the following is a factor of $x^4 + 1$ ?

(A) $x + 1$
(B) $x^2 + 1$
(C) $x^2 - \sqrt{2}x + 1$
(D) $x^2 + \sqrt{2}x - 1$
(E) None of these

31. The surface given by $z = x^2 - y^2$ is cut by the plane given by $y = 3x$, producing a curve in the plane. Find the slope of this curve at the point $(1, 3, -8)$.

(A) 3
(B) $-16$
(C) $-8\sqrt{\frac{2}{5}}$
(D) 0
(E) $\frac{18}{\sqrt{10}}$

32. If $A$ is an $n \times n$ matrix with diagonal entries, $a$, and other entries, $b$, then one eigenvalue of $A$ is $a - b$. Find another eigenvalue of $A$.

(A) $b - a$

(B) $nb + a - b$

(C) $nb - a + b$

(D) 0

(E) None of these

33. In ancient Egypt, the formula $A = \left(\frac{8d}{9}\right)^2$ was used for the area of a circle of diameter $d$. Using the correct multiple, relating the volume of a sphere to the area of a circle, what should the Egyptian formula be for the volume of the sphere of diameter $d$?

(A) $\frac{\pi d^3}{3}$

(B) $\left(\frac{8d}{9}\right)^3$

(C) $4\left(\frac{8}{9}d\right)^3$

(D) $\frac{2^8 \pi d^3}{3^5}$

(E) $\frac{2^7 d^3}{3^5}$

34. Find $k$ so that the matrix

$$A = \begin{pmatrix} k & 1 & 2 \\ 1 & 2 & k \\ 1 & 2 & 3 \end{pmatrix}$$

has eigenvalue $\lambda = 1$.

(A) $\frac{1}{2}$

(B) $-\frac{1}{2}$

(C) 0

(D) 1

(E) $-1$

35. Express $z^{14}$ in the form $a + bi$ if $z = \frac{(1+i)}{\sqrt{2}}$.

(A) $-i$

(B) $-1$

(C) 1

(D) $i$

(E) $\frac{(1+i)}{128}$

36. Which number is nearest a solution of $e^{\frac{-x}{100}} - e^{-x} = e^{-1}$ ?

(A) $1 - \frac{1}{2}\ln 4$

(B) $1 - \frac{1}{2}\ln 3$

(C) $1 - \frac{1}{2}\ln 2$

(D) $-1$

(E) 0

37. In the Laurent series for $f(z) = \frac{1}{(z-4)}$ centered at $z = 1$, th coefficient of $(z-1)^{-2}$ is

(A) 9

(B) $-9$

(C) $-3$

(D) 3

(E) $-1$

38. If $f(x) = \int_1^{x^2} \frac{dt}{1+t^3}$ then $f'(2)$ is

(A) $\frac{4}{65}$

(B) $\frac{1}{9}$

(C) $\ln\left(\frac{65}{2}\right)$

(D) $\ln\left(\frac{9}{2}\right)$

(E) 0.23

39. The series $\sum_{n=2}^{\infty} \frac{1}{n \cdot 3^n}$ must

(A) converge to a value greater than 1/4

(B) converge to a value greater than 1/9

(C) converge to a value less than 1/18

(D) converge to a value less than 1/12

(E) diverge

40. If $A$ is a square matrix of order $n \geq 4$ and $a_{ij} = i + j$ represents the entry in row $i$ and column $j$, then the rank of $A$ is always

(A) 1

(B) 2

(C) $n-2$

(D) $n-1$

(E) None of these

41. In the finite field, $Z_{17}$, the multiplicative inverse of 10 is

(A) 13

(B) 12

(C) 11

(D) 9

(E) 7

42. The system of equations $x^2 - y = a$ and $y - x = b$ has exactly one value of $x$ in its solution(s). This value of $x$ must be

(A) 0

(B) 1

(C) $\frac{3}{2}$

(D) $-\frac{1}{2}$

(E) $\frac{1}{2}$

43. $\log_4 64$ is identical to

(A) $\log_7 343$

(B) $\frac{\log_{10} 64}{\log_{10} 4}$

(C) $\log_8 256$

(D) Both (A) and (B)

(E) Both (A) and (C)

44. Let $S = \{\tan(k) \mid k = 1, 2, \ldots\}$. Find the set of limit points of $S$ on the real line.

(A) $(-\infty, \infty)$
(B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(C) $[0, \infty)$
(D) $(-\infty, 0]$
(E) The empty set

45. Given the sequence $a_n = \sin n$, $n \geq 1$, find $\lim_{n\to\infty} \inf\{a_n\} - \lim_{n\to\infty} \sup\{a_n\}$.

(A) 2
(B) 1
(C) 0
(D) $-1$
(E) $-2$

46. Find the length of a diagonal of a regular pentagon of side length 1.

(A) $2\cos\frac{\pi}{5}$
(B) $\sqrt{2}\left(1+\cos\left(\frac{\pi}{5}\right)\right)$
(C) $\sqrt{2}\left(1+\cos\left(\frac{2\pi}{5}\right)\right)$
(D) $\sqrt{2}$
(E) $\sqrt{2}+\sqrt{2\cos\left(\frac{2\pi}{5}\right)}$

47. Which of the following is a solution to the differential equatio
$y'' = 2y' + 8y$?

(A) $e^{2x} - e^{-4x}$
(B) $xe^{4x}$
(C) $e^{2x}$
(D) $e^{4x} - e^{-2x}$
(E) $xe^{-2x}$

48. Let $f(x)$ be defined and strictly increasing on $(a, b]$. Find the maximum value of $g(x) = -f(x)$ on $(a, b]$.

(A) $g(b)$
(B) $f(b)$
(C) $g(a)$
(D) $f(a)$
(E) Does not exist

49. Find the number of distinguishable permutations of six colored blocks if one is red, two are yellow, and three are blue.

(A) 360
(B) 60
(C) 720
(D) 120
(E) 240

50. Two vertices of an isosceles triangle are (1,2) and (4,6). The inradius of the triangle is 3/2. Find the maximum possible area for the triangle.

(A) $\frac{45}{4}$ (D) $\frac{9\sqrt{19}}{2}$

(B) $\frac{9\sqrt{19}}{4}$ (E) None of these

(C) $\frac{45}{2}$

51. The volume, $V$, of the region in space bounded above by the surface $x^2 + y^2 + z^2 = 4$ and below by $z = -\sqrt{x^2 + y^2}$ is represented by a triple integral in spherical coordinates as

$$\iiint_V \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

find the upper limit of integration for $\phi$.

(A) $\pi$ (D) $\frac{\pi}{4}$

(B) $\frac{3\pi}{4}$ (E) $\frac{\pi}{6}$

(C) $\frac{\pi}{2}$

52. The reliability of component $C$ is $R$. A system is designed as a series of $n$ subcomponents each of which is doubly redundant using two components, $C$. Find the reliability of the system.

(A) $R^n(1-R)^n$ (B) $R^n(2-R)^n$

(C) $R^{2n}$ (D) $\frac{R^n}{2^n}$

(E) $1-(1-R)^{2n}$

53. A family of curves is represented by the differential equation $ydx - xdy = 0$. Which of the following best describes the family of orthogonal trajectories to this given family?

(A) all parabolas with vertex at (0, 0)

(B) all hyperbolas with centers at (0, 0)

(C) all lines through (0, 0)

(D) all circles with centers at (0, 0)

(E) all lines parallel to the $y$-axis

54. Which of the following sequences is a solution for the difference equation $x_{n+2} + 36n = x_{n+1} + 6x_n$ ?

(A) $3^n + 6n$ (D) $3^n + 6 + n$

(B) $2^n + 6n + 1$ (E) $(-1)^n 2^{n+1} + 6n + 1$

(C) $6^n + 6n$

55. Let $f$ be a mapping from a topological space $\underline{\overline{X}}$ <u>onto</u> itself. Which of the following is true for continuous $f$?

(A) Every open set in $\underline{\overline{X}}$ is the image of an open set in $\underline{\overline{X}}$

(B) $f^{-1}(B)$ is bounded for each bounded set $B$ in $\underline{\overline{X}}$

(C) $f$ is one-to-one

(D) Both (A) and (B)

(E) Both (A) and (C)

56. At the point $(2, -1, 2)$ on the surface $z = xy^2$, find a direction vector for the greatest rate of decrease of $z$.

(A) $\hat{i} - 2\hat{j}$

(B) $\hat{i} - 4\hat{j}$

(C) $\dfrac{(\hat{i} - 4\hat{j})}{\sqrt{17}}$

(D) $-\hat{i} + 4\hat{j}$

(E) $\hat{i} + \hat{j}$

57. Let $G$ be a polyhedron with 27 vertices and 40 edges. Find the number of faces on $G$.

(A) 12

(B) 13

(C) 14

(D) 15

(E) Cannot be decided based on the given information

58. Find the coefficient of $x^2y^3$ in the binomial expansion o $(x - 2y)^5$.

(A) $-160$

(B) 80

(C) $-80$

(D) 8

(E) $-8$

59. In the evaluation of the integral $\int \dfrac{dx}{x(2+3x)^2}$, the coeffi cient of $\ln|(2 + 3x)|$ is

(A) $-\dfrac{3}{4}$

(B) $-\dfrac{1}{4}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{1}{4}$

(E) $-\dfrac{1}{36}$

60. A biased coin is tossed repeatedly until the first "tail" occu The expected number of tosses required to produce the first t is estimated as $T$. Assuming this is true, find the probability at least two tails in $3T$ tosses.

(A) $\dfrac{T^{3T} - (T-1)^{3T-1}(4T)}{T^{3T}}$

(B) $\dfrac{T^{3T} - (T-1)^{3T-1}(3T)}{T^{3T}}$

(C) $\dfrac{T^{3T} - (T-1)^{3T-1}(3T-1)}{T^{3T}}$

(D) $\dfrac{T^{3T} - (T-1)^{3T-1}(4T-1)}{T^{3T}}$

(E) None of these

61. The vertices of a quadrilateral are $(0,0), (1,4), (3,2)$, and $(5,5)$. Find the first coordinate of the centroid of the region.

(A) $\frac{9}{4}$

(B) $\frac{13}{6}$

(C) $\frac{7}{3}$

(D) $\frac{17}{6}$

(E) $\frac{8}{3}$

62. Let $R$ be a ring and let $x \neq 0$ be a fixed element in $R$. Which of the following is a subring of $R$ ?

(A) $\{ r \mid xr = 0 \}$

(B) $\{ x \mid x^{-1} \text{ exists in } R\}$

(C) $\{ x^n \mid n = 1, 2, \dots \}$

(D) $\{ nx \mid n \text{ is an integer}\}$

(E) Both (A) and (D)

63. If $f(x) = \ln x$, find

$$\lim_{m \to 0}\left[\lim_{n \to 0} \frac{f(2+m+n) - f(2+m) - f(2+n) + f(2)}{mn}\right]$$

(A) $\frac{1}{4}$

(B) 1

(C) $-1$

(D) $\frac{1}{2}$

(E) $-\frac{1}{4}$

64. Let $S = \{ x_1, x_2, \dots, x_n, \dots \}$ be a topological space where the open sets are $U_n = \{ x_1, \dots, x_n \}, n = 1, 2, \dots$ . Let $E = \{ x_2, x_4, \dots, x_{2k}, \dots \}$. Find the set of cluster points of $E$ .

(A) $S - \{x_1, x_2\}$

(B) $\{x_1\}$

(C) $\{x_2\}$

(D) $E - \{x_2\}$

(E) $S - E$

65. A stiff beam on two supports that are 20ft. apart is loaded by two uniform blocks with dimensions and weights as shown below.


How much of the total weight is supported at the left support?

(A) 150 (D) 100

(B) 120 (E) None of these

(C) 110

66. Given that $\sum_{n=1}^{\infty} a_n$ converges to $L$, which conclusion is valid for $\sum_{n=1}^{\infty} a_n^2$ ?

(A) It may diverge

(B) It converges absolutely

(C) It converges to $M < L$

(D) It converges to $M > L$

(E) It converges to $L^2$

# GRE MATHEMATICS TEST II

## ANSWER KEY

| | | | | |
|---|---|---|---|---|
| 1. | B | 23. | B | 45. |
| 2. | A | 24. | C | 46. |
| 3. | A | 25. | D | 47. |
| 4. | E | 26. | D | 48. |
| 5. | D | 27. | E | 49. |
| 6. | B | 28. | B | 50. |
| 7. | E | 29. | E | 51. |
| 8. | C | 30. | C | 52. |
| 9 | B | 31. | C | 53. |
| 10. | E | 32. | B | 54. |
| 11. | A | 33. | E | 55. |
| 12. | B | 34. | A | 56. |
| 13. | E | 35. | A | 57. |
| 14. | E | 36. | B | 58. |
| 15. | D | 37. | D | 59. |
| 16. | E | 38. | A | 60. |
| 17. | D | 39. | D | 61. |
| 18. | E | 40. | B | 62. |
| 19. | D | 41. | B | 63. |
| 20. | B | 42. | E | 64. |
| 21. | B | 43. | D | 65. |
| 22. | B | 44. | A | 66. |