GRE MATHEMATICS
TEST I

TIME: 2 hours and 50 minutes
66 Questions

DIRECTIONS: Choose the best answer for each question and mark the letter of your selection on the corresponding answer sheet.

1. The graph of the arccosine function is the graph of the arcsine function

   (A) translated horizontally $\pi / 2$ units to the right

   (B) first reflected in the horizontal axis and then translated vertically $\pi / 2$ units upward

   (C) first translated horizontally $\pi / 2$ units to the left and then reflected in the horizontal axis

   (D) first translated vertically $\pi / 2$ units downward and then reflected in the vertical axis

   (E) translated horizontally $\pi / 2$ units to the left

2. If $f(x) = e^x - e^{-x}$, then $[f'(x)]^2 - [f(x)]^2$ equals

(C) $2e^{-x}$ (E) $2e^{x}$

(D) 2

3. The domain of $f(x) = \dfrac{\sqrt[3]{x+2}}{x-6}$ is given by

(A) $(6, +\infty)$ (D) $[-2, +\infty)\setminus\{6\}$

(B) $[-2, +\infty)$ (E) $R\setminus\{6\}$

(C) $R\setminus\{-2, 6\}$

Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$. The determinant of the adjoint of M is

(A) 9 (D) 18

(B) 6 (E) 3

(C) 27

The number of generators of a cyclic group of order 8 is

(A) 6 (D) 2

(B) 4 (E) 1

(C) 3

6. An integrating factor for the ordinary differential equation $\dfrac{-2y}{x}\,dx + \left(x^2 y\cos y + 1\right)\,dy = 0$ is

(A) 1 (D) $-2x$

(B) $\dfrac{-2}{x}$ (E) x^2

(C) $\dfrac{1}{x^2}$

7. Assuming convergence, find $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}$

(A) $\frac{1}{2}(\sqrt{5} + 1)$

(B) $\frac{1}{2}(\sqrt{13} - 1)$

(C) $\frac{1}{2}(\sqrt{5} - 1)$

(D) $\frac{1}{2}(\sqrt{13} + 1)$

(E) $\frac{1}{2}(\sqrt{13} - \sqrt{5})$

8. Let x be a random variable possessing the probability density function

$$f(x) = \begin{cases} cx & x \in [0, 10] \\ 0 & \text{otherwise} \end{cases}$$

where $c \in R$. The probability that x is an element of $[1, 2]$ is

(C) $\frac{5}{100}$

(D) $\frac{7}{100}$

(E) $\frac{9}{100}$

9. Let the random variable X have the probability density function

$$f(x) = \begin{cases} 1 - \frac{x}{2} & x \in (0,2) \\ 0 & \text{otherwise} \end{cases}$$

The expected value of the random variable X^2 is

(A) $\frac{1}{3}$

(B) $\frac{5}{6}$

(C) $\frac{1}{2}$

(D) $\frac{1}{6}$

(E) $\frac{2}{3}$

10. Find the number of solutions of the set of all algebraic equations of height two.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

11. Define a metric on $R^2 = R \times R$ by $d[(x_1, y_1); (x_2, y_2)] = |x_2 - x_1| + |y_2 - y_1|$. The unit ball $d[(0,0); (x,y)] < 1$ is

(A) the interior of a circle with center (0, 0) and radius 1

(B) (0, 0)

(C) the interior of a square with vertices (–1, 1), (1, 1), (1, –1) and (–1, –1)

(D) the interior of a square with vertices (–1, 0), (0, 1), (1, 0) and (0, –1)

(E) the interior of a triangle with vertices (–1, –1), $(0, \sqrt{3})$, and (1, –1)

12. Which of the following is not equal to $f(x) = \frac{x+1}{x-1}$ when both are defined?

(A) $-f(x^{-1})$

(B) $[f(-x)]^{-1}$

(C) $f^{-1}(x)$

(D) $f^{-1}(x^{-1})$

(E) $\frac{1}{2}[f^{-1}(x) - f(x^{-1})]$

13. Let $M = \begin{bmatrix} 6 & 10 \\ -2 & -3 \end{bmatrix}$. The trace of M^5 equals

(C) 5^3

(D) $6^5 + (-3)^5$

(E) 33

14. The degree of the minimum polynomial satisfied by a nonscalar, 8 by 8, idempotent matrix M is

(A) 4

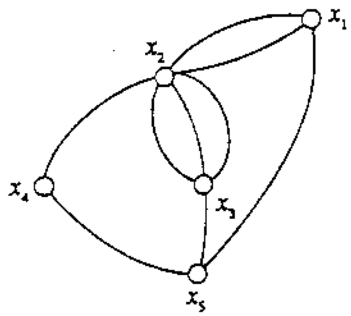
(B) 2

(C) 8

(D) 3

(E) 6

15. Find the incidence matrix for the graph:



(A) $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 2 & 0 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 1 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

16. The fixed point(s) of the Mobius transformation $w(z) = \dfrac{z-2}{z-1}$ is (are)

(A) $1 \pm \sqrt{3}$

(B) $1 \pm 2i$

(C) $2i$

(D) $1 \pm i$

(E) $-1 \pm \sqrt{2}\, i$

17. The sum of the 9^{th} roots of unity is

(A) 0

(B) 1

(C) 9

(D) 10

(E) $1 + i$

18. Let x , y , z represent Boolean variables. Which of the following is not a Boolean function?

(A) $f(x, y) = x\sqrt{y}$

(B) $f(x, y, z) = \max\{x, y, z\}$

(C) $f(x, y) = x^2 + y - xy$

(D) $f(x, y, z) = x + y + z - xy - yz$

(E) $f(x, y, z) = xyz$

19. What is the maximum perimeter of all rectangles that can be inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

(A) $4\sqrt{a^2 + b^2}$

(B) $\frac{8}{\sqrt{a^2 + b^2}}$

(C) $2\sqrt{a^2 + b^2}$

(D) $a^2 + b^2$

(E) $2(a^2 + b^2)$

20. Which of the following is a topological property?

(A) boundedness

(B) being a Cauchy sequence

(C) completeness

(D) being an accumulation (limit) point

(E) length

21. The value of $I = \oint_c \frac{\cos z}{z(z - \pi)}\, dz$ where C is the circle $|z - 1| = 2$ is

(A) 0

(B) $2i$

(C) $-2i$

(D) $-4i$

(E) $4i$

22. Let $p(x)$, $q(x)$, and $r(x)$ be open statements relative to the set S. Then $\sim(\exists x \in S)\,[(p(x) \vee q(x)) \wedge r(x)]$ is equivalent to

(A) $(\forall x \in S)\,\{[(\sim p(x)) \vee (\sim q(x))] \vee [\sim r(x)]\}$

(B) $(\forall x \in S)\,\{[(\sim p(x)) \wedge (\sim q(x))] \vee [\sim r(x)]\}$

(C) $(\forall x \in S)\,\{[(\sim p(x)) \wedge (\sim q(x))] \vee [r(x)]\}$

(D) $(\forall x \in S)\,\{[(\sim p(x)) \vee (\sim q(x))] \wedge [\sim r(x)]\}$

(E) $(\forall x \in S)\,\{[(\sim p(x)) \vee (\sim q(x))] \wedge [r(x)]\}$

23. Let $x_n = \frac{[illegible]}{n!}$ for $n = 1, 2, 3, \ldots$. Then $\lim_{n \to +\infty} \frac{[illegible]}{x_n}$ equals

(A) $\sqrt{e}$ (D) e^2

(B) e (E) e^{-1}

(C) $\sqrt{e^3}$

24. The derivative of $f(x) = \int_x^0 \frac{\cos xt}{t}\, dt$ is

(A) $-\frac{\cos 2x^2}{x}$ (D) $-\frac{\sin 2x^2}{x}$

(B) $\frac{1}{x}[1 + 2\cos x^2]$ (E) $\frac{\cos x^2}{x}$

(C) $\frac{1}{x}[1 + 2\sin x^2]$

25. The maximum value of the directional derivative on the surface $z = f(x, y) = xe^{xy} + y\cos x$ at $P(0, 1)$ is

(A) 1 (D) $\sqrt{4}$

(B) $\sqrt{2}$ (E) $\sqrt{5}$

(C) $\sqrt{3}$

26. The number of ordered partitions of the positive integer 5 is

(A) 20 (D) 14

(B) 18 (E) 12

(C) 16

27. The Wronskian of $f_1(x) = x^2 \sin x$ and $f_2(x) = x^2 \cos x$ is

(A) x^2 (D) $-x^4$

(B) $-x^2$ (E) $2x^4$

(C) x^4

28. Given the linear second-order difference equation

$$y_{k+2} - y_{k+1} - 2y_k = 0 \; ; \; k = 0, 1, 2, \ldots$$

$$y_0 = 9 \; ; \; y_1 = -12$$

find y_6.

(A) -54 (D) 27

(B) 64 (E) 54

(C) -32

29. Which of the following numbers is divisible by 9?

(A) 7224466
(B) 9224466
(C) 3224466
(D) 5224466
(E) 1224466

30. The inflection point for $f(x) = \frac{\ln x}{x}$ occurs at $x =$

(A) $\sqrt{e}$
(B) e
(C) $\sqrt{e^3}$
(D) e^{-1}
(E) $\sqrt{e^{-1}}$

31. The dimension of the null space of

$$M = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ -1 & 0 & 1 & 3 \end{bmatrix}$$

is:

(A) 2
(B) 1
(D) 3
(E) 0

32. Find the radical of the commutative ring Z_8.

(A) Z_8
(B) $\{0\}$
(C) $\{0, 2, 4, 6\}$
(D) $\{0, 2, 4\}$
(E) $\{0, 2\}$

33. Which of the following is equivalent to $\sin^3 x \cos^2 x$?

(A) $\frac{1}{16}[2\sin x - \sin 3x - 2\sin 5x]$
(B) $\frac{1}{16}[\sin x - 2\sin 3x - \sin 5x]$
(C) $\frac{1}{16}[2\sin x - \sin 3x - \sin 5x]$
(D) $\frac{1}{16}[2\sin x + \sin 3x - \sin 5x]$
(E) $\frac{1}{16}[\sin x + \sin 3x - \sin 5x]$

34. The absolute maximum of $f(x) = \cos 2x - 2\cos x$ on $[0, 2\pi]$ occurs at $x =$

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{2}$
(C) π
(D) $\frac{5\pi}{3}$
(E) $\frac{3\pi}{4}$

5. The eigenvalue which corresponds to the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for $M = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$ is

(A) 1

(B) 4

(C) -1

(D) -4

(E) 2

6. Evaluate the sum $\sum_{n=1}^{m} \arctan\left(\frac{1}{n^2+n+1}\right)$

(A) m^2+1

(B) $\frac{1}{m^2+m}$

(C) $\cot(m+1) - \frac{1}{m^2+1}$

(D) $\arctan(m+1) - \frac{\pi}{4}$

(E) $(-1)^m \sin(m+1) + \tan m$

. The conjugates of an element are the other roots of the irreducible polynomial of which the given element is a root. The conjugates of $\sqrt{\sqrt{3}+1}$ over the field of rational numbers are

(A)

(B) $\sqrt{\sqrt{3}+1}\,,\ -\sqrt{\sqrt{3}+1}$

(C) $\pm\sqrt{1+\sqrt{3}}\,,\ \pm\sqrt{1-\sqrt{3}}$

(D) $\pm\sqrt{\sqrt{3}+1}\,,\ \pm\sqrt{\sqrt{3}-1}$

(E) $\sqrt{\sqrt{3}+1}\,,\ -\sqrt{\sqrt{3}-1}$

38. The smallest positive integer n for which the inequality $2^n > n^2$ is true for $\{n, n+1, \ldots\}$ is

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

39. Consider the two player (P_1, P_2) game G with payoff matrix

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \end{array}$$

The minimax value of G is

(A) $\frac{5}{3}$

(B) 1

(D) 5

(E) 0

40. The first Newton approximation x_1 for a zero of $f(x) = x^3 - 2x$ with initial approximation $x_0 = 2$ is

(A) $\frac{12}{5}$

(B) 2

(C) $\frac{8}{5}$

(D) $\frac{6}{5}$

(E) $\frac{7}{5}$

41. The value of $\int_1^4 |x-2|\,dx$ is

(A) 3

(B) $\frac{5}{2}$

(C) 2

(D) $\frac{3}{2}$

(E) $\frac{7}{2}$

42. A Sylow 3-subgroup of a group of order 72 has order

(A) 3

(B) 9

(C) 18

(D) 27

(E) 36

43. Which of the following sets, together with the given binary operation *, does <u>not</u> form a group?

Note: Z = integers
Q = rationals
R = reals
C = complex numbers

(A) $G = \{a + b\sqrt{2} \in R\setminus\{0\} \mid a, b \in Q\}$
*: usual multiplication of real numbers

(B) $G = \{a + bi\sqrt{2} \in C\setminus\{0\} \mid a, b \in Q\}$
*: usual multiplication of complex numbers

(C) $G = \{\sqrt[3]{a} \in R \mid a \in Z\}$
*: for $a, b \in G$, $\sqrt[3]{a} * \sqrt[3]{b} = \sqrt[3]{a+b}$

(D) $G = R\setminus\{0\}$
*: for $a, b \in G$, $a * b = |a|\,b$

(E) $G = \{z \in C \mid |z| = 1\}$
*: usual multiplication of complex numbers

44. Let $M = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. Then $M^6 = kM$ for $k =$

(A) 2^6

(B) 2^8

(C) 2^{10}

(D) 2^{12}

(E) 2^{14}

45. From a group of 15 mathematics graduate school applicants, 10 are selected at random. Let P be the probability that 4 of the 5 applicants who would make the best graduate students are included in the 10 selected. Which of the following statements is true?

(A) $0 \leq P \leq \frac{1}{5}$

(B) $\frac{1}{5} < P \leq \frac{2}{5}$

(C) $\frac{2}{5} < P \leq \frac{3}{5}$

(D) $\frac{3}{5} < P \leq \frac{4}{5}$

(E) $\frac{4}{5} < P \leq 1$

46. The equation $r = 2\sin\theta - \cos\theta$ in rectangular coordinates is given by

(A) $x^2 + y^2 + x - 2y = 0$

(B) $x^2 - x + 2y = 0$

(C) $x^2 + y^2 + 2x - y = 0$

(D) $x^2 - y^2 - x + 2y = 0$

(E) $y^2 - x^2 - x + 2y = 0$

47. The decimal $2.0259\overline{259}$ is equivalent to which of the following?

(A) $\frac{20237}{9990}$

(B) $\frac{547}{270}$

(C) $\frac{20239}{9999}$

(D) $\frac{747}{370}$

(E) $\frac{737}{380}$

48. The general term of the Maclaurin series for xe^{-x^2} is

(A) $\frac{(-1)^n x^{2n}}{(n+1)!}$

(B) $\frac{(-1)^{n+1} x^{2n+1}}{n!}$

(C) $\frac{(-1)^n x^{2n+1}}{(n+1)!}$

(D) $\frac{(-1)^n x^{2n+1}}{n!}$

(E) $\frac{(-1)^{n+1} x^{2n}}{n!}$

49. The solution set for the inequality $x - \frac{3}{x} > 2$ is given by

(A) $(0, +\infty)$

(B) $(3, +\infty)$

(C) $(-1, 0) \cup (3, +\infty)$

(D) $(-\infty, 0) \cup (3, +\infty)$

(E) $(-\infty, 3)$

50. The volume (in cubic units) generated by rotating the region defined by the curves

$$y = x$$
$$y = 2\sqrt{x}$$

around the x-axis is

(A) $\frac{16\pi}{5}$

(B) $\frac{32\pi}{15}$

(C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

(E) π

51. Let A and B be subsets of U and denote the complement of subset X of U by X^c. Find $[[A \cap (A \cap B^C)] \cap B]^C$.

(A) B^c (D) U

(B) A^c (E) $\emptyset$

(C) $A \cup B^c$

52. The cross product $\vec{u} \times \vec{v}$ of the vectors

$$\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$$

$$\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$$

is given by

(A) $7\vec{i} + \vec{j} + 3\vec{k}$ (D) $-5\vec{i} + 5\vec{j} + 5\vec{k}$

(B) $5\vec{i} + 5\vec{j} + 3\vec{k}$ (E) $7\vec{i} - \vec{j}$

(C) -3

53. Find the number of left cosets of the cyclic subgroup generated by (1, 1) of $Z_2 \times Z_4$ where Z_n denotes the cyclic group of $\{0, 1, 2, \ldots, n-1\}$ under addition modulo n.

(A) 1 (D) 6

(B) 2 (E) 8

(C) 4

54. Up to isomorphism, how many abelian groups are there of order 36?

(A) 1 (D) 12

(B) 4 (E) 18

(C) 9

55. If $i = \sqrt{-1}$, then $\sum_{j=0}^{10} (-i)^j$ is

(A) i (D) $1 + i$

(B) -1 (E) $1 - i$

(C) $-i$

6. The set of gaussian integers, $R = \{a + ib \mid a, b \in Z \text{(integers)}\}$, is a commutative subring of the complex numbers. An element $u = e + id$ in R is a unit of R if there exists $V \in R$ such that $uv = 1$. The unit(s) of R is (are)

(A) ± 1 (D) $\pm 1, \pm i$

(B) $\pm i$ (E) 1

(C) $1, i$

7. The number of solutions (equivalence classes) of the congruence $3x + 11 \equiv 20 \pmod{12}$ is:

(A) no solutions (D) 4

(B) 1 (E) 6

(C) 3

8. Let R be the region defined by

$$y = x - 1; \quad x = 1; \quad y = -x + 3$$

Find the maximum value of $f(x, y) = -2x + 3y$ on R.

(A) -2 (D) 4

(B) 1 (E) -1

(C) 2

59. If $|x|$ is large, then $f(x) = \dfrac{x^5 - x^4 + x^3 + x}{x^3 - 1}$ is approximately

(A) $x^2 + x$ (D) $x^2 + 1$

(B) $x^2 - x + 1$ (E) $x^2 - x$

(C) x^2

60. The number of vertices of an ordinary polyhedron with 12 faces and 17 edges is

(A) 7 (D) 9

(B) 5 (E) 13

(C) 11

61. Let T be a linear transformation of the plane such that $T(1, 1) = (-1, 1)$ and $T(2, 3) = (1, 2)$. Then $T(2, 4)$ equals

(A) $(4, 2)$ (D) $(2, 4)$

(B) $(2, -4)$ (E) $(-3, 2)$

(C) $(3, -2)$

62. The function $f(z) = \sin x \cosh y + v(x, y)i$ is analytic for $v(x, y)$ equal to

(A) $\cos x \cosh y$

(B) $\cos x \sinh y$

(C) $-\sin y \cosh x$

(D) $\sin x \sinh y$

(E) $\sin y \cosh x$

63. Let T represent a nonsingular linear transformation from E^n into E^n. Which of the following is <u>not</u> true?

(A) Null space of $T = \{0\}$

(B) T is one-to-one

(C) Dimension of null space is zero: $\text{Dim}\, N(T) = 0$

(D) Dimension of range space is n: $\text{Dim}\, R(T) = n$

(E) $\text{Dim}\, N(T^{-1}) = \text{Dim}\, R(T)$

64. Find the value of the sum: $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots$

(A) $\frac{11}{3}$

(B) $\frac{5}{2}$

(C) 2

(D) 3

(E) ∞

65. The circles

$$c_1: x^2 + y^2 + 2ax + 2by + c = 0$$

$$c_2: x^2 + y^2 + 2a'x + 2b'y + c' = 0$$

are orthogonal if

(A) $2aa' + 2bb' = c + c'$

(B) $a + a' + b + b' = cc'$

(C) $aa' - bb' = c - c'$

(D) $2aa' - 2bb' = c - c'$

(E) $a + b + c = a' + b' + c'$

66. The inverse of the matrix $M = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ is the matrix

$M^{-1} = \frac{1}{6}\begin{bmatrix} -7 & 8 & a \\ 8 & -10 & -4 \\ 4 & b & -2 \end{bmatrix}$ where

(A) $a = 5 ; b = -2$

(B) $a = 3 ; b = 2$

(C) $a = 1 ; b = -3$

(D) $a = 2 ; b = -3$

(E) $a = 2 ; b = 3$

GRE MATHEMATICS TEST I

ANSWER KEY

1.	B	23.	B	45.	B
2.	A	24.	A	46.	A
3.	E	25.	B	47.	B
4.	E	26.	C	48.	D
5.	B	27.	D	49.	C
6.	C	28.	A	50.	D
7.	D	29.	C	51.	D
8.	B	30.	C	52.	D
9	E	31.	B	53.	B
10.	B	32.	C	54.	B
11.	D	33.	D	55.	C
12.	D	34.	C	56.	D
13.	E	35.	C	57.	C
14.	B	36.	D	58.	D
15.	C	37.	C	59.	B
16.	D	38.	E	60.	A
17.	A	39.	A	61.	A
18.	D	40.	C	62.	B
19.	A	41.	B	63.	E
20.	D	42.	B	64.	D
21.	C	43.	D	65.	A
22.	B	44.	C	66.	A

GRE MATHEMATICS TEST I

DETAILED EXPLANATIONS OF ANSWERS

1. (B)

Whereas the fundamental identity for the trigonometric functions is $\sin^2 x + \cos^2 x = 1$, the fundamental identity for the inverse trigonometric functions is $\arcsin x + \arccos x = \pi/2$. Thus $\arccos x = \pi/2 - \arcsin x$. The curve of $\arcsin x$ reflected in the horizontal axis will represent the curve of $-\arcsin x$. Adding $\pi/2$ is geometrically equivalent to translating the curve vertically $\pi/2$ units upward.

2. (A)

Since $f'(x) = e^x + e^{-x}$, we have

$$\begin{aligned}[f'(x)]^2 - [f(x)]^2 &= [e^x + e^{-x}]^2 - [e^x - e^{-x}]^2 \\ &= e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x} \\ &= 4\end{aligned}$$

Using the identities

$f(x) = 2\sinh x$, $f'(x) = 2\cosh x$, and $\cosh^2 x - \sinh^2 x = 1$,