SOLUTION GUIDE TO MATH GRE FORM GR8767

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The questions for this solution guide can be found <u>here</u>.

Solution 1. (B) Any three (non-colinear) points uniquely determines a plane. Since the xz-plane contains these points, that's the answer.

Solution 2. (C) I is certainly true if a and b are both positive or both negative, but if $a = -1$ and $b = 1$ (e.g.) we have $-1 < 1$ and the same relation holds for the reciprocals. II is certainly not true if c is negative. III is true, luckily, since addition doesn't affect less than or greater than signs. This narrows down our answer to (C) already – we can verify that IV is true.

Solution 3. (B) A straightforward integral.
$$\int_0^1 \int_0^x xy \, dy \, dx = \int_0^1 \left(\frac{xy^2}{2} \right) \Big|_0^x dx = \int_0^1 \frac{x^3}{8} \, dx = \frac{x^4}{8} \Big|_0^1 = 1/8.$$

Solution 4. (A) Another easy calculus question. We obtain $x^e \cdot e^x + e^x \cdot e \cdot x^{e-1}$. Absorbing that e into the exponent gives us the answer.

Solution 5. (A) We can integrate both of these partial derivatives to obtain $x^2 + xy + f(y) + C$ and $xy + y^2 + g(x) + D$. Putting these together gives us $x^2 + xy + y^2 + C$.

Solution 6. (C) The slope is negative, then zero, then positive. It has discontinuities between those segments. That leaves only one option.

Solution 7. (A) The bottom graph is $|x|$ and the top graph is $x + 2$. We are integrating from -1 to 1. That leaves, again, one option.

Solution 8. (E) Let us look at the partial sums. We have $\frac{n}{n+1} = 1 - \frac{1}{n+1}$. Therefore the terms do not tend to 0 as $n \to \infty$. The sum cannot possibly converge.

Solution 9. (E) With repetitions allowed, this is easy. Every time, we must not pick 0. Therefore we have a 9/10 chance of not screwing up every time. This gives us $(9/10)^k$.

Solution 10. (C) We need to invert the matrix C and multiply our code on the right by C^{-1}. The easy way to invert a 2×2 matrix is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
That makes $C^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$. Then
$$\begin{pmatrix} 51 & -3 \\ 31 & -8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 54 & 45 \\ 39 & 15 \end{pmatrix}.$$
Dividing this matrix by 3 and putting the numbers in order gives 18-15-13-5, or ROME.

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Solution 11. (B) If $\sin^{-1} x = \pi/6$, then $x = 1/2$ (or something like that). Then $\cos^{-1}(1/2) = \pi/3$. Indeed, our answer should be the complement of $\pi/6$.

Solution 12. (B) This problem is disguising itself as something more difficult.

$$e^{\sin^2 x} e^{\cos^2 x} = e^{\sin^2 x + \cos^2 x} = e^1 = e.$$

That makes our integral $e \cdot \pi$.

Solution 13. (D) We see that the top approaches 16 and the bottom approaches 0 as $x \to 2$. The limit is therefore not finite, giving us only one answer.

Solution 14. (A) If the total use decreased by 5%, at least one portion of the city would have had to decrease by 5%. Therefore something said was in error.

Solution 15. (E) We can use $(1,1)$ and $(-1,0)$ as a basis for $\mathbb{R}^2$. We have that $(3,5) = 5(1,1) + 2(-1,0)$. Therefore $f(3,5) = 5 \cdot f(1,1) + 2 \cdot f(-1,0) = 5 + 4 = 9$.

Solution 16. (B) This sounds (roughly) like parabolic motion from p to q. That would make our graph concave down, and only one point satisfying the Mean Value Theorem for $[a,b]$.

Solution 17. (C) $*$ is certainly commutative. To check for an identity, we would need that there exists a b such that
$$a + b + 2ab = a$$
for any $a \in \mathbb{Q}$. The choice $b = 0$ works - it solves $b(1 + 2a) = 0$ for all $a \in \mathbb{Q}$. Finally, to look for an inverse, we would need a solution for $a * b = 0$.

$$a + b + 2ab = 0 \implies b(1 + 2a) = -a \implies b = \frac{-a}{1 + 2a}.$$

However, this only works if $a \neq -1/2$. If $a = -1/2$, then we would need to solve $-1/2 + b - b = 0$, which is impossible. Hence not every element has an inverse.

Solution 18. (D) If $abab = aabb$, then multiplying by a^{-1} on the left and b^{-1} on the right gives us $ba = ab$. Hence G is abelian.

Solution 19. (D) Taking a derivative, we get
$$f'(x) = e^x - c.$$
Hence we have a local extremum at $x = \log c$, which luckily exists as $c > 0$. This is seen to be a minimum, as $f'(0) < 0$ and $f(\log c + \varepsilon) > 0$.

Solution 20. (C) The only way for a polynomial to satisfy $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$ is for it to be constant. A polynomial of degree n is determined by $n + 1$ points, and we can see that $f(x) = f(x + n)$ for all $n \in \mathbb{N}$. Hence $f(x) = 11$.

Solution 21. (A) It is probably easiest to look at our options. We see that most options give us nothing but garbage. However,
$$e^{\log x/2} = e^{\log(x^{1/2})} = x^{1/2} = \sqrt{x}.$$

Solution 22. (B) This is a classic trigonometric integral. Let $x = \sin \theta$. Then $dx = \cos \theta \, d\theta$. We obtain
$$\int_0^{\sin y} \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta}{\sqrt{\cos^2 \theta}} \, d\theta = \int d\theta = \theta.$$

Switching back from θ to x, we obtain $\theta = \sin^{-1} x$, so this integral evaluates to

$$\sin^{-1} x \Big|_0^{\sin y} = y - 0 = y.$$

The next integral is easy.

$$\int_0^1 y \, dy = 1/2.$$

Solution 23. (C) This statement is saying that S is provable by induction. If we know that $S(n_0)$ is not true, then no $n < n_0$ can be true. If it were, then by induction so would anything higher than it, including $S(n_0)$. However, we can say nothing about $S(n_0 + 1)$ or higher statements.

Solution 24. (D) Let us just start working this out.

$$g(1) = 3 \cdot g(2) = 3 \cdot 3 \cdot f(3) = 3 \cdot 3 \cdot 2 \cdot f(2) = 3 \cdot 3 \cdot 2 \cdot 2 \cdot f(1) = 36.$$

Solution 25. (D) If we know that $3x + 7y \equiv 0$, then multiplying this quantity by 5 will still yield something divisible by 11. That gives us $15x + 35y$. Of course we know that $11x + 44y$ is divisible by 11, so we take

$$(15x + 35y) - (11x + 44y) = 4x - 9y \equiv 0.$$

That's the good solution. Here's another option: it's not hard to come up with that $(10, 2)$ is a solution. Plugging in, (D) and (E) are the only options that work. But even without realizing the extent of the manipulations we can do as described above, multiplying any solution by a constant still gives us a solution, e.g. $(5, 1)$ works too. But now that doesn't work for (E), leaving only (D). Inductive reasoning can be a very good method if you don't exactly know how a problem should be solved!

Solution 26. (E) The topologist's sine curve ($k = 0$) is a classic example of a strange connected space. Most of the space is path connected, but there is no path between $(0, 0)$ and the rest of the graph. The space is nonetheless connected because any neighbourhood of the graph near $x = 0$ must include the point $(0, 0)$. The same would hold if k is anything in the range $[-1, 1]$ – that is where $\sin x$ is infinitely densely oscillating around $x = 0$. Anything outside that range would cause a disconnection however.

Solution 27. (A) We need $f(1)$ and $f'(1)$ to agree at both definitions, so we want $f(1) = 1$ and $f'(1) = 1$. That means that $f(1) = a + b + c = 1$ and $f'(1) = 2a + b = 1$. The second equation tells us that $b = 1 - 2a$, and hence $c = a$. a can be any nonzero real number.

Solution 28. (E) The semicircle described has a radius of $(b - a)/2$. Then the area under this curve is $1/2 \cdot ((b - a)/2)^2$. Working this out is $(b - a)^2 \cdot \pi/8$.

Solution 29. (C) If we multiply by 3, we end up with a curve with the same endpoints $(a, 0)$ and $(b, 0)$. However, the height of the curve at its maximum is $3(b - a)/2$. Therefore we no longer have a semicircle, but an ellipse – its major axis is thrice its minor axis.

Solution 30. (A) By translation, we might as well assume that $a = -r$ and $b = r$. That would make $f(x) = \sqrt{r^2 - x^2}$. This is easier to manage, and we obtain

$$f'(x) = \frac{-x}{\sqrt{r^2 - x^2}} \implies f(x)f'(x) = -x.$$

We can also see that this doesn't depend on r, which is convenient. Then our integral is

$$\int_{-r}^{r} f(x)f'(x)\,dx = \int_{-r}^{r} -x\,dx = 0.$$

Alternatively (and probably more easily), we can use integration by parts. Let $u = f(x)$ and $dv = f'(x)dx$. Then

$$\int_{a}^{b} f(x)f'(x)\,dx = f(x)^2 - \int_{a}^{b} f(x)f'(x)\,dx \implies 2\int_{a}^{b} f(x)f'(x)\,dx = f(b)^2 - f(a)^2.$$

We see that we need to just calculate $f(b)^2/2 - f(a)^2/2$. But of course $f(a) = f(b) = 0$, so the integral is zero.

Solution 31. (B) L'Hopital's rule should deal with this easily enough.

$$\lim_{x \to \pi} \frac{e^{-\pi} - e^{-x}}{\sin x} = \lim_{x \to \pi} \frac{e^{-x}}{\cos x} = \frac{e^{-\pi}}{-1} = -e^{-\pi}.$$

Solution 32. (B) The first two vectors are linearly dependent, and we can throw out the zero vector. The first, third, and fourth vectors are linearly independent. But the fifth vector is a linear combination of those three. That gives us 3 linearly independent vectors.

Solution 33. (E) Certainly $y > x^2$ needs to be in there somewhere. We also want to have $1/x < y$, but that's not quite it – on the negative side that's not satisfied. Instead if we use $xy < 1$ that takes the positive side and the negative into account. For $x < 0$, we can take any $y > 0$ we want.

Solution 34. (A) It would be easiest to draw out this problem. Draw the problem as written, then draw the mirror image of the person behind the mirror. Because the mirror is flat on the wall, they'll both be height t. Draw the line of sight from the (real) eyes of the person to the mirror image's feet. This gives us two similar triangles: one with hypotenuse given by eyes-to-mirror and the other with hypotenuse eyes-to-feet. The first triangle has height $t - h$ and base d; the second triangle has height t and base $2d$. This means that $2(t - h) = t$, from which we conclude $2h = t$. The distance d is irrelevant.

Solution 35. (B) Let's a look at the column space. The difference of the first two columns is $(1, 1, 1, 1, 1)$. Adding this to the second row gives the third, and again the fourth and again the fifth. Therefore only two columns are linearly independent.

We could also take a look at the row space and notice that the difference of any two adjacent rows is $(5, 5, 5, 5, 5)$ and conclude the same.

Solution 36. (A) We need to minimise the equation $d(x, y) = x^2 + y^2$ with the constrict that $y = 8/x$. Putting this in,

$$d(x) = x^2 + 64/x^2 \implies d'(x) = 2x - 128/x^3.$$

Solving this for $d'(x) = 0$,

$$2x = 128/x^3 \implies 2x^4 = 128 \implies x^4 = 64 \implies x = 2\sqrt{2}.$$

That makes $y = 8/(2\sqrt{2}) = 2\sqrt{2}$ as well. Therefore $d(x, y) = 8 + 8 = 16$, so the shortest distance is 4.

Solution 37. (D) Not all the steps are reversible. (3) implies (4) because it is the special case $y = 0$. But (4) does not imply (3) in general – why would it?

Solution 38. (A) Let's see what happens if we start with small powers – there should be a pattern.

$$M^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M^3 = I_3.$$

As such, $M^{100} = M^{99} \cdot M = (I_3)^{33} M = M$.

Solution 39. (B) For $x > 0$, we have $f(x) = 1$. For $x < 0$, we have $f(x) = -1$. This makes $f(x)$ symmetric around the origin – an odd function, $-f(x) = f(-x)$. As such, any symmetric integral is zero.

Solution 40. (C) Note first that

$$\int xe^x \, dx = xe^x - e^x.$$

We can rewrite this to be a little more tenable:

$$y' + \frac{y}{x} = e^x.$$

We need an integration factor $\mu(x)$ such that $\mu(x)/x = \mu'(x)$. We can see that $\mu(x) = x$ fits the bill. Using the general theory of linear ODEs[1], we get

$$y(x) = \frac{\int xe^x \, dx + C}{x} = e^x - \frac{e^x}{x} + \frac{C}{x}.$$

Since $y(1) = 0$, we have that $C = 0$. That makes $y(2) = e^2 - e^2/2 = e^2/2$.

Solution 41. (D) We have that $y' = -e^{-x} + 1$, so $y'(1) = 1 - 1/e \approx 2/3$. Both (C) and (D) look about this slope, so we should look towards concavity. We have $y'' = e^{-x}$, so the graph is concave up everywhere. That means we must pick (D).

Solution 42. (D) We need the angle between $(1, 0, 1)$ and $(0, 1, 1)$. This can be obtained by

$$|\vec{a} \cdot \vec{b}| = |a| \cdot |b| \cdot \cos\theta.$$

We see that $(1, 0, 1) \cdot (0, 1, 1) = 1$, and $|a| \cdot |b| = 2$. Therefore $\cos\theta = 1/2$, making $\theta = \pi/3$.

Solution 43. (E) We know that the complex roots must show up in conjugate pairs. As such, the other two roots should be $2 - i$ and $1 + i$. We know that the negation of the sum of the roots is the coefficient of x^3 and the positive product of the roots is the constant term. Therefore we expect $-6x^3$ and 10 as the constant term. That leaves one option.

Solution 44. (B) If we consider the linear change of variables $x_0 \mapsto x_0 + h$, we obtain the equal limit

$$\lim_{h \to 0} \frac{f(x_0 + 2h) - f(x_0)}{h}.$$

By the usual business, this is equal to $2f'(x_0)$.

Solution 45. (E) We should use the ratio test. This gives us

$$\lim_{n \to \infty} \left| \frac{e^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n x^n} \right| = \lim_{n \to \infty} \frac{e}{n+1} \cdot |x|.$$

This has an infinite radius of convergence, as any choice of x will make this limit zero.

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[1]Read up more here

Solution 46. (E) Taking the log of both sides (for the sake of argument), that gives us $\log x \log y = \log y \log x$. Therefore for $x, y > 0$, every point is valid. Moreover, if $x \le 0$ or $y \le 0$, the logarithms are not defined, so the function does not exist. Therefore we obtain the first quadrant, but not the axes.

Solution 47. (D) Since there are eight points in S, the power set $P(S)$ has $2^8 = 256$ elements. Therefore any particular set need not be a member of $\mathfrak{G}$. (A) and (E) require that, and (B) and (C) require the absence of certain sets. However, since there are 8 singleton sets in S and only 6 sets missing from $\mathfrak{G}$, two singletons must be in $\mathfrak{G}$.

Solution 48. (D) Since it is important, let us calculate (in abstract) ST:
$$(ST)(p(x)) = S(x \cdot p(x)) = x \cdot p'(x) + p(x) = (TS + \mathrm{id})(p(x))$$
Therefore $ST - TS = \mathrm{id}$.

Solution 49. (C) The existence of such a subgroup means that G contains an element of order 7. That makes our choices (B), (C), or (E) – the order of the group must be divisible by 7. However, G must not have any elements of even order. There are two ways to see this. One way is the following theorem: if p is a prime number dividing the order of a group G, then G has an element of order p.

As another idea[2]: we can break up the entire group as follows
$$G = \bigcup \{g, g^{-1}\}$$
taken over an appropriate amount of representatives of $g \in G$. Since no (non-identity) element of the group is its own inverse, each of the sets $\{g, g^{-1}\}$ has two elements but for $\{e\}$. That means we have written $|G| = 2n + 1$, more or less, meaning that the order of G has to be odd. Combine that with divisibility by 7 as above.

Solution 50. (D) We have to add up infinitely many possibilities, but it shouldn't be too bad. The game must end on an odd turn. The game ending on the first term has probability $1/2$. To end on the third turn, we first must flip two tails then one heads. That has a probability of $1/8$. Similarly, ending on the fifth turn has a probability of $1/32$. That means we must sum
$$\sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{4}\right)^n.$$
The total sum of this is $\frac{1/2}{3/4} = 2/3$.

Alternatively, let p be the probability that the first player wins. Then if the first flip comes up tails, we effectively start a new game and the second player has a probability p of winning. But the total probability of the second player winning is $1 - p$, hence
$$1 - p = \frac{1}{2} \cdot p \implies p = \frac{2}{3}.$$

Solution 51. (E) If this series converges, then its limit x will satisfy
$$\sqrt{3 + 2x} = x \implies 3 + 2x = x^2 \implies x^2 - 2x - 3 = 0 \implies (x - 3)(x + 1) = 0.$$
This gives the options -1 or 3. Since our answer shouldn't be negative (as x starts out positive), we take $x = 3$.

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[2]Adapted from <u>here</u>

Solution 52. (C) Calculating the eigenvalues is not too difficult.

$$\det \begin{bmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{bmatrix} = (5 - \lambda)^2 - 1 = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6).$$

That gives us the bigger eigenvalue.

Solution 53. (B) We know that $\dim V = 4$, as it has basis $\{1, x, x^2, x^3\}$. For W, we know that any $p(x) \in W$ must have $x(x-1)(x+1)$ as a factor. Since the maximal degree of any polynomial in W is 3, we know that W is a linear subspace generated by $x(x-1)(x+1)$. That makes $\dim W = 1$.

Solution 54. (E) For φ to be a homomorphism, we must have $\varphi(e) = e$ and $\varphi(xy) = \varphi(x)\varphi(y)$. We see that $\varphi(e) = a^3$, so we must have $a^3 = e$. Moreover, in that case $\varphi(xy) = axya^2 = axa^2 \cdot aya^2 = \varphi(x)\varphi(y)$. In that case, φ is a homomorphism.

Solution 55. (C) Since we are being asked about extrema, let us take some partial derivatives.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y, \quad \frac{\partial f}{\partial y} = 3y^2 + 3x.$$

The first equation gives us $y = -x^2$. Replacing this into the second equation, we have

$$3(-x^2)^2 + 3x = 0 \implies x^4 + x = 0 \implies x(x^3 + 1) = 0 \implies x = 0, -1.$$

This makes the set of solutions $(0,0)$ and $(-1,-1)$. We now need to figure out what sort of points these are. To calculate the Hessian, we need the second derivatives:

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = 3.$$

Therefore the determinant of the Hessian is $D(x,y) = 36xy - 9 = 9(4xy - 1)$. Then $D(0,0) < 0$ and $D(-1,-1) > 0$, making the first a saddle point and the second a maximum or a minimum. Since $f_{xx}(-1,-1) < 0$, that gives us a maximum.

Solution 56. (A) We are looking for the next term in the Taylor series. We have the second-order polynomial, so we need the third term. Hence the error is given by

$$\frac{K(x-a)^3}{3!}$$

where K is a maximum of $|f^{(3)}(x)|$ on $[1, 1.01]$. First, we need that, for $f(x) = x^{1/2}$,

$$f'(x) = \frac{1}{2} \cdot x^{-1/2}, \quad f''(x) = \frac{-1}{4} \cdot x^{-3/2}, \quad f^{(3)}(x) = \frac{3}{8} \cdot x^{-5/2}.$$

Then K is achieved at $x = 1$, so $K = 3/8$. That makes our error term

$$\frac{3/8 \cdot (0.01)^3}{6} = \frac{1}{16} \cdot 10^{-6}.$$

However, we still need to establish whether this should be positive or negative. Looking at the Taylor series, we see that it is alternating. This means that the Taylor polynomial is an overestimate if and only if the subsequent term is negative. Since the third term of the Taylor series is positive (as we calculated above), the function $p(1.01)$ underestimates $\sqrt{1.01}$ and thus the error is positive.

Solution 57. (C) We know that if the first character must be a digit and the last a digit, that gives $N_1 = 10$ and $N_2 = 100$. That narrows our options down to (A) or (C), but (A) is clearly wrong since it doesn't have anything related to the four signs (like, for example, the digit 4 appearing in the answer). Thus we have only one reasonable option.

Solution 58. (B) Let $f(z) = u(x, y) + i \cdot v(x, y)$. Then f satisfies the Cauchy-Riemann equations, so $u_x = v_y$ and $u_y = -v_x$. If $f(z)$ has no imaginary component, then $v(x, y) = 0$. Therefore $u_x = 0$ and $u_y = 0$ as well, so $u(x, y)$ is a constant function. Therefore the imaginary axis – as well as everything else – maps to a single point.

Another possible solution: recall that the open mapping theorem states that a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is open, so it sends open sets to open sets. In particular, $\mathbb{C} \subset \mathbb{C}$ is open, but $\mathbb{R} \subset \mathbb{C}$ (the image of f) is not open. Therefore this map cannot be non-constant and holomorphic, but it is holomorphic (since it is analytic). Therefore it must be constant.

Solution 59. (D) In a confusing way, this question is asking us what the approximate total change in $f(x)$ is over the interval. First, the function increases by 2, then it decreases by 5, increases by 7, and decreases by 2. Adding those up gives 16.

Solution 60. (C) We want at least 10 more than the expected value of sixes rolled. The standard deviation of a binomial distribution is given by $\sigma = \sqrt{np(1-p)}$, where p is the probability of the event and n the number of trials. For us, $p = 1/6$. Therefore

$$\sigma = \sqrt{360 \cdot 1/6 \cdot 5/6} = \sqrt{50} \approx 7.$$

Our value of 70 falls between 1 and 2 standard deviations of the mean, which means it falls within the 84th and 97.5th percentiles. That gives us a probability between 0.025 and 0.16.

Solution 61. (C) We can see that an option for A is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Given that 'No unique answer' is not an answer choice, we would guess that the answer is zero. If we want to be more explicit, we use the fact that $A^2 = I_2$ and compute symbolically:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}$$

so that both $b(a + d) = c(a + d) = 0$. The trace is $a + d$, so if the trace is not equal to zero, then we must have $b = c = 0$. The rest of the matrix implies that $a^2 = d^2 = 1$, so we have $a, d = \pm 1$. We have already thrown out the case where $a = d = 1$ and $a = d = -1$, so we are back into the guess we made above.

Solution 62. (B) This makes the most sense in the case of Green's theorem. Since B is a counterclockwise-oriented boundary, we would have

$$\int_B 3y \, dx + 4x \, dy = \iint_S 4 - 3 \, dx \, dy = A$$

where A is the area of S. That would make our answer 1.

Solution 63. (D) There is a theorem that, if $f(a) = c$ and $f(b) = d$,

$$\int_a^b f(x)\,dx + \int_c^d f^{-1}(y)\,dy = bd - ac.$$

For us, we use that $f(0) = 1$. Moreover, since the integral $\int_0^\infty f(x)\,dx$ is finite, we know that $\lim_{x\to\infty} f(x) = 0$. That gives us that

$$\int_0^\infty f(x)\,dx + \int_1^0 f^{-1}(y)\,dy = 0.$$

Subtracting the integral and reversing the limits gives the answer.

Another way to see this: take the graph of $f(x)$ on $[0, \infty)$ and flip it across the line $x = y$. Then we obtain the graph of $f^{-1}(y)$ but only on $[0, 1]$. This is a quick and dirty way to solve this problem without relying on arcane knowledge like the above.

Solution 64. (C) I'm not certain what 'bounded' means in this case, so we skip this condition. Continuous is the weakest condition, and it is also sufficient. Suppose that $\{U_i\}$ is an open cover of T. Since f is onto, we can pull this cover back to a cover $\{V_i = f^{-1}(U_i)\}$ of S. By compactness we have a finite subcover $V_1, \ldots, V_n$ of S, which means that $U_1, \ldots, U_n$ covers T. Therefore T is compact.

To quickly discount (D), a bijective function of topological spaces need not respect the topology at all. For instance, let $S = T$ but endow S with the indiscrete and T with the discrete topologies, and let $f\colon S \to T$ be the identity. Then S is compact but T is decidedly not.

Solution 65. (A) Factoring this, we get $(y' + y)^2 = 0$. Solving the differential equation $y' = -y$ gives us $y = Ce^{-t}$. We are looking for an exponential decay graph – or the upside-down version of one, as we see in (A).

One way to rule out (D) and (E) directly is that the solution curves intersect; the fundamental theorem on existence and uniqueness of solutions to differential equations means that different solutions can't share any points. Otherwise, if we try to write down the solution with initial value $(0, 0)$, (D) and (E) imply that there are two different options, violating uniqueness.

Solution 66. (B) I is the ring $\mathbb{Q}[\sqrt{2}]$, a subfield of $\mathbb{R}$ even. II gives us $\mathbb{Z}[1/3]$, the localisation of $\mathbb{Z}$ at the element 3, which is still a ring. III has some strange boundedness condition which makes it not a ring – it's not closed under addition, in particular.