Isolated Complex Singularities and Their CR Links

Dedicated to Professor Sheng Gong on the occasion of his 75th birthday

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1. Introduction: In this paper, we survey some of the recent studies on the complex structure of an isolated complex singularity through the CR structures of its local links. We also present some examples and open problems, which may guide us for the future investigation along these lines of research.

To start with, we let V be a complex space with an isolated non-regular point $p \in V$. Assume $\dim_{\mathbf{C}} V \geq 2$. One of the main themes in several complex variables is to understand the complex structure of the germ of V at p. This problem can be approached by both algebraic and analytic methods. The algebraic method uses the ideal of holomorphic functions defining the germ of V at p, while the analytic method relies more on the so-called CR and subelliptic analysis on the local links of V near p. Our survey in this article will mainly be on the analytic aspect of such a study.

2. Links: Since our concern here is mainly local, we assume that V is embedded into a complex Euclidean space $\mathbf{C}^m$ with $p = 0$. Suppose $\rho(z)$ be a smooth strongly plurisubharmonic function defined in a certain connected open subset U in $\mathbf{C}^m$ with $V \subset\subset U$, $\rho(0) = 0$, $\rho(z) > 0$ for $z \neq 0$ and $\rho : U \to \text{Image}(\rho(U))$ proper. The following definition is standard in the literature (See [Md] [Mil]):

Definition 2.1: Let $M_{\rho,\epsilon} = \{z \in V : \ \rho(z) = \epsilon\}$. The $M_{\rho,\epsilon}$ is called the ϵ-link of $(V, 0)$ associated with ρ. When $\rho = |z|^2$, $M_{\rho,\epsilon}$ is called the standard ϵ-link of V. For the standard link, we simply write M_ϵ for $M_{\rho,\epsilon}$.

By the Sard theorem, for almost all $0 < \epsilon << 1$, $M_{\rho,\epsilon}$ is a compact smooth submanifold embedded in V of real co-dimension 1. Also, for the standard $\rho = |z|^2$, there is an $\epsilon_0 < 1$ such that for any $0 < \epsilon << \epsilon_0$, $M_{\rho,\epsilon}$ is smooth. In what follows, we will assume $\epsilon << 1$, unless otherwise stated. Also, we assume $\dim_{\mathbf{C}} V, \ \ \dim_{\mathbf{C}} V' \geq 2$.

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A crucial fact that makes it possible to study $(V, 0)$ through its links is that $M_{\rho,\epsilon}$, whenever smooth, carries a partial complex structure naturally induced from V, called the inherited Cauchy-Riemann structure. (See [BER]). More precisely, for any $q \in M_{\rho,\epsilon}$, define

$$T_q^{(1,0)} M_{\rho,\epsilon} = T_q^{(1,0)} V \cap CT_q M_{\rho,\epsilon}.$$

Then the complex dimension of $T_q^{(1,0)} M_{\rho,\epsilon}$ is $\dim_{\mathbf{C}} V - 1$ for any q. Moreover, $T_q^{(1,0)} M_{\rho,\epsilon}$ depends smoothly on $q \in M_{\rho,\epsilon}$ and thus naturally defines a complex smooth vector bundle $T^{(1,0)} M_{\rho,\epsilon}$ over $M_{\rho,\epsilon}$ with $T_q^{(1,0)} M_{\rho,\epsilon}$ as its fiber space over $q \in M_{\rho,\epsilon}$. Moreover, there is a nowhere zero smooth real vector field T over $M_{\rho,\epsilon}$ such that we have the following splitting:

$$CTM_{\rho,\epsilon} = T^{(1,0)} M_{\rho,\epsilon} + T^{(0,1)} M_{\rho,\epsilon} + \mathbf{C}T.$$

This splitting defines what we call the inherited CR structure over $M_{\rho,\epsilon}$. The CR structure is strongly pseudoconvex by our assumption that ρ is strongly plurisubharmonic.

Next, we assume $(V', 0)$ be another germ of complex space with an isolated singularity at 0. ρ' is defined as before. We can then similarly define the link $M'_{\rho',\epsilon'}$. We say that $M_{\rho,\epsilon}$ is CR equivalent to $M'_{\rho',\epsilon'}$ if there is a smooth diffeomorphism F from $M_{\rho,\epsilon}$ to $M'_{\rho',\epsilon'}$ that respects the CR structures defined above. Namely,

$$F_* \left(T^{(1,0)} M_{\rho,\epsilon} \right) = T^{(1,0)} M'_{\rho',\epsilon'}.$$

A fundamental result along these lines of research is the following classical Hartogs type extension theorem (see [AG] [Wh], etc):

Theorem 2.2: Suppose 0 is the only isolated singularity of the normal complex space V and V', respectively. Assume that F is a CR equivalence map from $M_{\rho,\epsilon}$ to $M'_{\rho',\epsilon'}$. Write $V_\epsilon = \{z \in V : \rho(z) < \epsilon\}$ and $V'_{\epsilon'} = \{z \in V' : \rho'(z) < \epsilon'\}$. Then F extends as a biholomorphic map from V_ϵ to $V'_{\epsilon'}$. In particular, $(V, 0)$ is holomorphically equivalent to $(V', 0)$.

Unfortunately, the above reduction is not reversible. By the Chern-Moser theory [CM], for $\epsilon \neq \epsilon'$, $M_{\rho,\epsilon}$ and $M_{\rho,\epsilon'}$ have, in general, very different CR structures. Finding the invariants from the links which are directly related to the complex structure of the singularities is always an interesting problem in this field of Several Complex Variables.

3. Spherical links: The unit sphere in $\mathbf{C}^n$ with $n \geq 2$ may be regarded as the simplest strongly pseudoconvex CR manifold. However, it does not bound any normal singularity.

Compact spherical CR manifolds may be regarded as the simplest strongly pseudoconvex CR manifolds which may bound normal singularities. Here, we recall that a CR manifold is said to have a spherical CR structure if it is locally CR equivalent to a piece of the sphere of the same dimension. A typical example of spherical CR manifolds is a spherical space form (in terms of [Da]): $M_\Gamma := \partial \mathbf{B}^n/\Gamma$, where $\mathbf{B}^n$ is the unit ball in $\mathbf{C}^n$ and Γ is a finite subgroup of $Aut(\mathbf{B}^n)$ with 0 as its only fixed point. By the invariant polynomial theory, one can construct an algebraic realization Ψ of $\mathbf{C}^n/\Gamma$ into $\mathbf{C}^N$ for a certain $N \geq n$ such that $\Psi(\partial\mathbf{B}^n/\Gamma)$ bounds a normal isolated complex singularity with $\Psi(\partial\mathbf{B}^n/\Gamma)$ as one of its algebraic link. (See [Ca] [Fo]). As an explicit example, let $V := \{(x, y, z) \subset \mathbf{C}^3 : \quad y^2 = 2xz\}$. Then its standard link M with $\epsilon = 1$ is algebraic and spherical, for the holomorphic map $(t, \tau) \to (t^2, \sqrt{2}t\tau, \tau^2)$ is a holomorphic covering map from $\partial\mathbf{B}^3$ into M. Conversely, we have the following:

Theorem 3.1([HJ]): Suppose that ρ is algebraic and has no critical value from 0 to ϵ_0. Suppose that for a certain $\epsilon < \epsilon_0$, $M_{\rho,\epsilon}$ carries a compact spherical CR structure. Assume that $V \subset \mathbf{C}^n$ with $n \geq 2$ has only an isolated normal singularity at 0. Then there is a finite unitary subgroup $\Gamma \subset Aut(\mathbf{B}^n)$ and a biholomorphic map from $\mathbf{B}^n/\Gamma$ to $V_\epsilon := \{z \in V : \quad \rho(z) < \epsilon\}$. In particular, $(V, 0)$ is holomorphically equivalent to the quotient singularity $(\mathbf{B}^n/\Gamma, 0)$.

Theorem 3.1 is not stated in [HJ]. But the proof there with a slight modification gives Theorem 3.1, which we discuss as follows.

Proof of Theorem 3.1: For any point $w \in M_{\rho,\epsilon}$, by the assumption, there is a CR equivalence map Φ_w from a piece of the sphere $\mathbf{S}$ to a piece of the link $M_{\rho,\epsilon}$ near w. Since ρ is algebraic strong pseudoconvex, by the algebraicity theorem of the author proved in [Hu], Φ_w is also an algebraic map. In particular, we see that $M_{\rho,\epsilon}$ can be locally defined by (real Nash) algebraic functions. Now, we fix one of Φ_w, denoted by Φ. Then Φ extends to an algebraic map (possibly multiple-valued) from $\mathbf{C}^n$ into $\mathbf{C}^m$. Let E be the set of poles and branching points of Φ. Then $\Phi(\mathbf{S} \setminus E) \subset M_{\rho,\epsilon}$. Let γ be a Jordan curve in $\mathbf{S}$ with $\gamma([0, c_0)) \subset M_{\rho,\epsilon} - E$ and $\gamma(c_0) \in E$. Suppose that for a certain sequence $\{t_j\}$ with $t_j < c_0$ and $t_j \to c_0$, $\Phi(t_j) \to q$. Let A_q be an affine complex subspace of dimension n such that there is an affine linear map π_q fixing q, projecting $\mathbf{C}^m$ to A_q, with $\pi_q|_V$ biholomorphic near q. Write $\Phi^q = \pi_q \circ \Phi$ and $N^q = \pi_q(M_{\rho,\epsilon})$ near q. Then Φ^q, N^q must also be (Nash) algebraic. Then with the same argument as in the proof of Lemma 3.1 of [HJ], we see that a similar statement as in Lemma 3.1 of [HJ] holds. This then implies that $\Phi^q(\gamma(t))$ has limit q as $t \to c_0^-$. Now, the same argument as in §4 of [HJ] shows that Φ^q extends holomorphically across $\gamma(c)$ and thus along any path inside $\mathbf{S}$. Since $\mathbf{S}$ is simply connected, we conclude that the extension of

Φ, still denoted by Φ, gives a single-valued holomorphic covering map from $\mathbf{S}$ to $M_{\rho,\epsilon}$. Let $\Gamma := \{\sigma_j\}_{j=1}^k$ be the deck transformations associated with this covering. We see easily that $\Gamma = \{\sigma_j\}$ extends to a finite subgroup of $Aut(\mathbf{B}^n)$ such that Φ induces a biholomorphic map from $\mathbf{B}^n/\Gamma$ to V_ϵ. Since Γ has no fixed point over $\mathbf{S}$, we can assume without loss of generality that 0 is the only fixed point of Γ. Now, the rest of the proof follows from Theorem 2.2. ■

From the proof of theorem 3.1, we also see the following:

Corollary 3.2: Let V be a complex analytic space embedded in $\mathbf{C}^m$ with only an isolated singularity at 0. Suppose that for a certain algebraic ρ as before, the corresponding ϵ-link $M_{\rho,\epsilon}$ carries a (compact) spherical CR structure. Then $M_{\rho,\epsilon}$ is CR equivalent to a CR spherical space form $\partial\mathbf{B}^n/\Gamma$ with $\Gamma \subset Aut(\mathbf{B}^n)$ a certain finite group with the only fixed point at 0. In particular, the fundamental group of $M_{\rho,\epsilon}$ is isomorphic to Γ.

Corollary 3.3: Let M be a (Nash) algebraic strongly pseudoconvex spherical CR submanifold in $\mathbf{C}^m$. Then M is CR equivalent to a CR spherical space form $\partial\mathbf{B}^n/\Gamma$ with $\Gamma \subset Aut(\mathbf{B}^n)$ a finite group with the only fixed point at 0.

Corollary 3.3 fails when ρ is assumed to be real analytic. Indeed, by an example of Burns-Shnider [BS], there are real analytic spherical CR submanifolds of dimension 3 in $\mathbf{C}^2$ with fundamental group of infinite order.

Moreover, in their private conversations with the author, Siu and Burns suggested the following more general construction of analytic spherical links with fundamental group of infinite order, by using the Grauert tube technique. This thus shows that the algebraicity assumption in the above results are crucial for the statements to hold:

Let M be a complex manifold with a Hermitian metric h. The Grauert tube over its holomorphic tangent bundle, induced from h, is defined to be the domain $\Omega := \{v \in T^{(1,0)}M : h(v,v) < 1\}$. Assume that M is a ball quotient $\mathbf{B}^n/\Gamma$, where $\Gamma \subset Aut(\mathbf{B}^n)$ is a (fixed-point free) lattice. Assume that h is a hyperbolic metric with a negative constant holomorphic sectional curvature. (Such an (M,h) is called a hyperbolic space form([Mok]).) The following result is classical in the literature.

Proposition 3.4: Assume that (M,h) is a compact hyperbolic space form of dimension $n \geq 1$. Then the Grauert tube of its holomorphic tangent bundle is a domain in $T^{(1,0)}M$ with real analytic spherical boundary.

Proof of Proposition 3.4: Indeed, for any point $q \in M$, let U_q be a small neighborhood of q in M and choose a z-coordinates system in U_q with $z(q) = 0$ such that h takes the following standard form in the z-coordinates:

$$h = \sum h_{j\bar{l}} dz_j \overline{dz_l} : \quad h_{j\bar{l}} = c_n \frac{(1-|z|^2)\delta_{jl} + \overline{z_j} z_l}{(1-|z|^2)^2}.$$

Here c_n is a constant depending only on n, which we can assume to be 1. Use (z, ξ) for the coordinates of $\pi^{-1}(U_q) \subset T^{(1,0)}M$. In these coordinates, Ω and its boundary are then defined, respectively, by

$$\Omega \cap \pi^{-1}(U_q) := \{(z, \xi) \in z(U_q) \times \mathbf{C}^n : \ \sum h_{j\bar{l}}(z)\xi_j \overline{\xi_l} < 1\} \ \text{ and}$$

$$\partial\Omega \cap \pi^{-1}(U_q) = \{(z, \xi) \in z(U_q) \times \mathbf{C}^n : \ \sum h_{j\bar{l}}(z)\xi_j \overline{\xi_l} = 1\}.$$

Then the negativity of the holomorphic bisectional curvatures guarantees the strong pseudoconvexity of $\partial\Omega$ and the CR symmetry of $\partial\Omega$ guarantees the spherical property of $\partial\Omega$.

In fact, for any point $(z_0, \xi_0) \in \partial\Omega$, let $\sigma \in Aut(\mathbf{B}^n)$ be such that $\sigma(z_0) = 0$ and σ pulls back the vector ξ_0 to a vector of the form $v_0 = (a_1, 0, \cdots, 0)$ with $a_1 > 0$. Then $(\sigma, d\sigma)$ defines a CR diffeomorphism from a neighborhood of (z_0, ξ_0) in $\partial\Omega$ to a neighborhood of $(0, v_0)$ in $\partial\Omega$. Apparently, $a_1 = 1$. Now, Ω near $(0, v_0)$ is defined by an equation of the form $\rho = |\xi|^2 + 2|z|^2 + O(|(\xi - v_0)|^k |z|^l) < 1$ with $k + l \geq 3$. Hence, we see that Ω is strongly pseudoconvex and all boundary points are CR equivalent. Hence, $\partial\Omega$ is spherical. ∎

Now, applying the CR embedding theorem of Boutet de Monvel and Kohn [Bout] [Kn] we can find a real analytic CR embedding F from $\partial\Omega$ into a certain $\mathbf{C}^N$. The extension theorem of Kohn-Rossi [FK] shows that F extends to a holomorphic map from Ω into $\mathbf{C}^N$. Apparently F must be a local holomorphic embedding from $\Omega \setminus M$ and maps M into a point which we can assume to be 0. Making use of the Kohn-Rossi theorem, the Harvey-Lawson theorem and applying a normalization to resolve the normal-crossing singularities if necessary, we can assume, without loss of generality, that $F(\partial\Omega)$ bounds a normal Stein space with a unique isolated normal singularity at 0. F is biholomorphic from $\Omega \setminus M$ to its image and $F^{-1}(\{0\}) = M$. Based on this example, we pose the following:

Problem 3.5: Suppose V is a normal complex space with an isolated singularity at p. Suppose that for some $0 < \epsilon << 1$ and a real analytic strongly plurisubharmonic function ρ with $\rho(z) > \rho(0) = 0$ for $z \neq 0$, $M_{\rho,\epsilon}$ is a real analytic spherical CR manifold. What can we say about the complex structure of V at 0?

4. Unitary equivalence, Brieskorn spheres and Siu's program: In §3, we discussed the isolated singularities bounded by spherical CR manifolds. For general links, we have the following rigidity result proved in Ebenfelt-Huang-Zaitsev [EHZ]:

Theorem 4.1 ([EHZ]): Suppose $V, V' \subset \mathbf{C}^N$ be irreducible complex spaces with a singularity at 0, respectively. Suppose that $\dim_{\mathbf{C}} V = n \geq 4$ and $2N < 3n-1$. For ϵ, ϵ' write M_ϵ and $M'_{\epsilon'}$ for the standard ϵ-link and ϵ'-link of V and V', respectively. Assume that M_ϵ and $M_{\epsilon'}$ are smooth near p and p', respectively; and assume that there is a local CR equivalence map from a piece of M_ϵ near p to a piece of $M'_{\epsilon'}$ near p'. Then there is a unitary map $\mathcal{U}$ such that $\epsilon'\mathcal{U}(V) = \epsilon V'$. Namely, after a scaling, $(V, 0)$ and $(V', 0)$ are unitary equivalent.

The above result was first obtained by Webster in [We2] for the codimenional one case. Theorem 4.1 reveals that the CR structure of links well over-determine the complex structures of the singularities. While the codimensional restriction in Theorem 4.1 is important to get an equivalence for the singularities through the local CR equivalence, however we have not found so far an example which shows that the codimension restriction is also necessary in the global setting. Namely, we pose the following:

Problem 4.2: Suppose $V, V' \subset \mathbf{C}^N$ be complex spaces with isolated complex singularity at 0. Suppose for $\epsilon, \epsilon' = 1$, the standard links M_ϵ and $M'_{\epsilon'}$ are globally CR equivalent. What can we say about $(V, 0)$ and $(V', 0)$?

When the dimension of V is small, even the topological information of the links gives a strong implication of the complex structure near the singular point. A famous result of this type is the Mumford theorem [Md], which states that if V only has an isolated normal singularity at 0 and $\dim V = 2$, then any local smooth link $M_{\rho,\epsilon}$ can not be simply connected. For $n \geq 3$, Mumford's result fails and , one has the following famous Brieskorn spheres [Br] [Mil]:

Let $\vec{a} =< a_1, \cdots, a_n >$ with $n \geq 4$. Here $a_j's$ are positive integers, at least one of which is bigger than 1. Define

$$V(\vec{a}) = \{(z_1, \cdots, z_n) : \ \sum_{j=1}^{n} z^{a_j} = 0\}.$$

Then $0 \in V(\vec{a})$ is an isolated singularity. Write $M_\epsilon(\vec{a}) = \{z \in V(\vec{a}) : \ \ |z| = \epsilon\}$. By the work of Brieskorn and Milnor [Br] [Mil], for many choices of $\vec{a}$, $M_\epsilon(\vec{a})$ are the topological spheres, but may or may not be diffeomorphic to the standard sphere. In fact, all Milnor exotic spheres can be realized as the links of the above form.

Based on this phenomenon, Siu in [Siu4] has recently proposed a program trying to determine the complex structure of $(V, 0)$ by using the topological structure of its link together with some other geometric information. The reader is referred to [Siu4] for more details on this matter.

5. Holomorphic deformation of an isolated complex singularity: Links have been playing an important role in the study of the deformation theory of isolated complex singularities. (See, for instance, [BM] [Mi] [BE] [Lem] and references therein). Here, we only discuss some results obtained in the author's recent joint work with Luk-Yau in [HLY].

Let $\mathcal{V} \subset D(\subset \mathbf{C}^N)$ be a complex analytic variety with $0 \in \mathcal{V}$ as a singular point. Suppose that there is a holomorphic map $\pi : \mathcal{V} \to \Delta := \{t \in \mathbf{C} : |t| < 1\}$ such that $V_t := \pi^{-1}(t)$ is a complex analytic variety in D with only isolated complex singularities near 0 for each t. Then $\{V_t\}$ is a holomorphic family of complex spaces. V_t is said to be a holomorphic deformation of V_0. Define $\mathcal{M}_\epsilon := \{z \in \mathcal{V} : \ |z| = \epsilon\}$ and $M_{t,\epsilon} := \pi^{-1}(t) \cap \mathcal{M}_\epsilon$. Then we obtain a CR family of strongly pseudoconvex CR manifolds $\{M_{t,\epsilon}\}$ in case all links are smooth. More precisely, we have the following definition:

Definition 5.1: (X, π, Δ) is called a CR family of strongly pseudoconvex CR manifolds if the following holds: (i). X itself is a strongly pseudoconvex CR manifold; (ii) π is a surjective CR map and each fiber has a naturally inherited CR structure.

Making use of the Kuranishi-Akahori-Webster embedding theorem [Ku] [Ak] [We2] and the work of Siu-Ling [Siu1][Siu3][Ling] on the normal Stein completion of a family of $(1, 1)$ convex-concave manifolds, the following is derived in [HLY]:

Proposition 5.2: Suppose (X, π, Δ) is a CR family of strongly pseudo-convex CR manifolds. Assume that $\dim_{\mathbf{R}} X \geq 7$. Then there is a unique complex Stein 2-normal space $\widehat{X}$, which has X as part of its smooth boundary and a holomorphic map $\widehat{\pi}$ from $\widehat{X}$ to Δ such that $(\widehat{X}, \widehat{\pi}, \Delta)$ is a holomorphic family of Stein spaces with isolated singularities. Moreover $\widehat{\pi}$ has smooth boundary value π along X.

The above Stein space $\widehat{X}$ is called the Siu-Ling filling of X. Proposition 5.2 indicates a very nice correspondence between a CR family of strongly pseudoconvex CR manifolds and a holomorphic family of Stein spaces with isolated singularities. This makes it possible to study the holomorphic family of singular spaces by applying the method of subelliptic analysis. Before stating some results along these lines, we recall the following definitions of the Kohn-Rossi complex and Kohn-Rossi cohomology group [FK].

Let M be an oriented CR manifold of CR dimension $(n-1)$ and let T be a nowhere vanishing real vector field along M transversal to the contact bundle. Assign a pseudo-Hermitian metric $< \cdot, \cdot >$ over M with the following properties:

For any $p \in M$ and $v_1, v_2 \in T_p^{(1,0)}M$, $< v_1, \overline{v_2} >_p = 0$, $< v_1, T|_p >_p = 0$, $< \overline{v_1}, T|_p >_p = 0$, $< \overline{v_1}, \overline{v_2} >_p = \overline{< v_1, v_2 >_p}$.

Choose a real contact form θ over M and choose $\{\omega_1, \cdots, \omega_{n-1}\} \subset (T_p^{(1,0)})^*$ such that

$$\{\omega_1, \cdots, \omega_{n-1}, \theta|_p, \overline{\omega_1}, \cdots, \overline{\omega_{n-1}}\} \subset (CT_p)^*$$

is an orthonormal basis. Then any $(m+l)$-form ω at p can be uniquely decomposed as

$$\omega = \sum_{I,J:|I|+|J|=m+l} a_{I\overline{J}} \omega^I \wedge \overline{\omega^J} + \theta \wedge \cdots .$$

We define the projection map of type (m,l) $\pi_{(m,l)}|_p$ to be such that

$$\pi_{(m,l)}|_p(\omega) = \sum_{I,J:|I|=m,|J|=l} a_{I\overline{J}} \omega^I \wedge \overline{\omega^J}.$$

Denote the image set of $\pi_{(m,l)}|_p$ by $\Lambda_p^{(m,l)}(M)$. This linear space depends smoothly on p and thus defines a smooth vector bundle over M with smooth projection map $\pi_{(m,l)}$, the space of whose smooth sections over $U \subset M$ is denoted by $\Lambda^{(m,l)}(U)$. Then $d_b := \pi_{(m,l+1)} \circ d$ is a smooth differential operator of first order, mapping $\Lambda^{(m,l)}(U)$ into $\Lambda^{(m,l+1)}(U)$ for any open subset U of M. Here d is the usual DeRham differential operator over M. The integrability of the CR structure shows that $d_b^2 = 0$ and thus one can form a differential complex, called the Kohn-Rossi complex. The Kohn-Rossi cohomology of order $(0,k)$, $H_{KR}^{(0,k)}(M)$, is defined as the quotient of the space of closed $(0,k)$-forms with the space of exact $(0,k)$-forms. It is well known [FK] that for different choices of the pseudo-Hermitian metrics, one arrives at the isomorphic Kohn-Rossi cohomology groups.

The following CR extension theorem proved in [HLY] is useful to study the holomorphic family of complex spaces.

Theorem 5.3 ([HLY]): Suppose that (X, π, Δ) is a CR family of strongly pseudoconvex CR manifolds with $\dim X \geq 7$. Assume that X can be embedded to a complex manifold. Suppose that $\dim H_{KR}^{(0,1)}(M_t) = constant$. Then any smooth CR function f_0 over $M_0 = \pi^{-1}(0)$ can be extended as a smooth CR function f over X.

When applying a result of Catlin [Cat1-2], one sees that the assumption that X can be embedded to a complex manifold is satisfied automatically. The same remark applies in what follows. The proof of Theorem 5.1 is based on the work of Catlin on the mixed boundary value problem for the $\bar{\partial}$- problem, Siu-Ling's work on the direct image theorems for a family of $(1,1)$ convex-concave manifolds. As an immediate application, one has the following simultaneous CR embedding property under the constant dimensionality of the first Kohn-Rossi cohomology group of each fiber.

Corollary 5.4([HLY]): Under the assumption as in Theorem 5.1, if M_0 can be CR embedded into $\mathbf{C}^N$ by f_0, then the map $F = (f, \pi)$ embeds the family $\{M_t\}_{|t|<<1}$ into $\mathbf{C}^N \times \Delta$. Here f is the smooth CR extension of f_0 to X. In particular, M_t for $|t| << 1$ can be embedded into $\mathbf{C}^N$ near $f_0(X_0)$.

Corollary 5.4 is due to Tanaka [Ta] when f is only required to depend smoothly on the parameter t. In this setting, the family can also be assumed to be just a smooth family instead a CR family. ([Ta])

An immediate consequence of Corollary 5.4 is the flatness of the holomorphic family of the Stein space with the Siu-Ling filling as its total space. The proof of the following normality theorem in [HLY] is also based on the CR extension theorem in Theorem 5.2.

Theorem 5.5: Suppose the assumptions in Theorem 5.2 hold. Then $\widehat{X_0} = \widehat{\pi}^{-1}(0)$ is normal.

A special case of Theorem 5.5 is due to Fujiki in [Fu].

The just mentioned results in [HLY] can be used to study the simultaneous blowing-down problem for a family of complex manifolds with exceptional sets.

Recall that $\widetilde{M}$ is said to be a smooth strongly pseudoconvex complex manifold if $\widetilde{M}$ is a complex manifold with smooth boundary $\partial\widetilde{M}$, that is strongly pseudoconvex with respect to $\widetilde{M}$. Let $\widetilde{X}$ be a complex manifold with X as part of its strongly pseudoconvex boundary. We call $(\widetilde{X}, \widetilde{\pi}, \Delta)$ a family of smooth strongly pseudoconvex complex manifolds if (I): $\widetilde{\pi}$ is a surjective holomorphic map from $\widetilde{X}$ to Δ, which extends smoothly to $X = \cup_t \partial\widetilde{\pi}^{-1}(t) = \cup_t \partial\widetilde{X}_t$, where $\widetilde{X}_t = \widetilde{\pi}^{-1}(t)$; (II) (X, π, Δ) is a CR family of strongly pseudoconvex manifolds. Now, let f be a holomorphic map from $\widetilde{X}_0 := \widetilde{\pi}^{-1}(0)$ to the complex space $Y \subset \mathbf{C}^m$, that is biholomorphic near $\partial\widetilde{X}_0$ and extends to a smooth CR diffeomorphism from the boundary to its image. Also, assume that $f(\partial\widetilde{X}_0)$ bounds precisely the complex space Y, which has, at most, isolated singularities and has $f(\partial\widetilde{X}_0)$ as its smooth boundary. We say $\widetilde{X}_0$ resolves the singularities of Y through f in a broad sense (and in the regular sense if Y does have isolated singularities). Notice that f then must be biholomorphic from $\widetilde{X_0} \setminus E$ into $Y \setminus \mathrm{Sing}(Y)$ and proper from $\widetilde{X_0}$ to Y, where $\mathrm{Sing}(Y)$ is the singular set of Y. $E = f^{-1}(\mathrm{Sing}(Y))$ is called the generalized exceptional set of $\widetilde{X}$. (And E is the exceptional set in the regular sense if it does not have any connected components of codimension greater than one in $\widetilde{X}_0$.) We will call such an f a blowing-down map from $\widetilde{X}_0$ to its image Y. There have been many papers in the past on when a family of strongly pseudoconvex complex manifolds (with exceptional sets) can be simultaneously blown-down. (See the paper [Ri11-2] and the references therein.)

The above mentioned results can be used to give the following:

Theorem 5.6: Let $(\widetilde{X}, \widetilde{\pi}, \Delta)$ be a smooth holomorphic family of strongly pseudoconvex complex manifolds. Assume that $X = \cup_{t\in\Delta}\partial\widetilde{X_t}$ can be CR embedded into a complex manifold. Suppose that

$$\dim H^{(0,1)}_{KR}(\partial\widetilde{X}_t) = \text{constant}$$

and $\widetilde{X}_0$ is at least of complex dimension 3. Suppose that f_0 is a blowing-down map from $\widetilde{X}_0$ to $\mathbf{C}^m$. Then there is a map $F = (\widetilde{f}, \widetilde{\pi})$ from $\widetilde{X}_\eta := \widetilde{\pi}^{-1}(\Delta_\eta)$ to $\mathbf{C}^m \times \mathbf{C}$, which extends smoothly over $\cup_{|t|<\eta\ll 1}\partial\widetilde{X}_t$ such that $\widetilde{f}|_{\widetilde{X}_t}$ is a (holomorphic) blowing-down map from $\widetilde{X}_t$ to $\mathbf{C}^m$ with $\widetilde{f}|_{\widetilde{X}_0} = f_0$.

The argument in [HLY] does not cover the cases when the fiber is of three dimension. Here, we state the following open question:

Problem 5.7: Let $\{M_t\}_{t\in\Delta}$ be a CR family of 3-dimensional strongly pseudoconvex CR manifolds. Suppose that the total space X admits a normal Stein filling. Suppose that M_0 can be CR embedded into $\mathbf{C}^N$. Under what conditions, can be the nearby fiber M_t embedded into $\mathbf{C}^N$?

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