Description of Some of my Recent Research Work
Xiaojun Huang, September, 2011

My recent research work has focused on numerous problems in Complex Analysis of Several Variables which are also related to questions in non-linear analysis, partial differential equations, differential geometry and classical dynamics. More specifically, I have been working on the following topics:

(1). Normal form theory and holomorphic equivalence problem of real submanifolds in a complex manifold, holomorphic hulls of real submanifolds in complex spaces, the complex Plateau problem, as well as their interaction with celestial mechanics, KAM theory, etc;

(2). Rigidity problems in several complex variables, as well as their applications and interactions with super-rigidity problems in the theory of complex singularities, complex geometry, and classical dynamics;

(3). CR deformation of a compact strongly pseudoconvex CR manifold; simultaneous embedding of a CR family of strongly pseudoconvex CR manifolds; the fiber normality problem for a Stein filling of a (1,1) convex-concave family.

(4). Chern-Moser-Weyl tensor and applications to the embedding and unique continuation problem for CR maps.

My research on these problems has been continuously supported by the National Science Foundation, and my most two recent research grants (DMS-1101481, DMS-0801056) were funded by the NSF with the highest priority. The following paragraphs contain a more detailed discussion of some of my research activities with an emphasis mainly on work in the past six years:

Papers cited below are posted in my home web page:
www.math.rutgers.edu/~huangx

1. Normal forms, geometry and dynamics for real submanifolds in $\mathbb{C}^n$ with CR singular points— Solution to the problems of Moser, Moser-Webster, Kenig-Webster and Bishop: The classical Bishop problem (proposed by Bishop in 1965) asks if a smooth real $n$-manifold ($n > 1$) in $\mathbb{C}^n$ bounds a unique smooth ‘minimal manifold’ (with $M$ as part of its smooth boundary), that is smoothly foliated by holomorphic disks, near an elliptic CR singular point where the Levi-curvature is positive in a certain sense. The Bishop problem is of geometric nature and can be regarded as a local version of the complex Plateau problem. It can also be formulated as the boundary and interior regularity problem for a certain degenerate elliptic PDE. Moser-Webster, in their celebrated paper, studied the normal form theory for such manifolds near a non-exceptional CR singular point. Motivated by the results obtained by Moser-Webster and himself, Moser formulated in 1985 two problems for a real surface in $\mathbb{C}^2$ with an elliptic CR singular point with symmetry, which is also called an elliptic CR singular point with a vanishing Bishop invariant. The Bishop problem and the Moser problems have attracted much attention since they were raised, because of their importance at least due to the following reason: (a) the ‘minimal manifold’ provides the holomorphically
invariant local hull of holomorphy, (b) the work of Gromov, Eliashberg, Hofer and others reveals a close relationship of the study for such problems with problems in Symplectic Geometry and Topology, and (c) through the work of Moser-Webster, one sees their profound connection with problems in celestial mechanics, KAM theory and classical dynamics. In [Hu1], I was able to answer several problems left open by Kenig-Webster. This, combined with a series of papers of Kenig-Webster, and Moser-Webster, finally provided a complete solution to the longstanding open question that Bishop asked in 1965.

The first problem of Moser mentioned above concerns the flattening for a real analytic Bishop surface with a vanishing Bishop invariant, and was answered in the affirmative in Huang-Krantz [HK]. The second problem of Moser is related to the classification of Bishop surfaces with elliptic complex tangency. After the work of Moser-Webster and Moser, the remaining case is the vanishing Bishop invariant case.

Recall that a Bishop surface $M \subset \mathbb{C}^2$ with a vanishing Bishop invariant at $0 \in M$ is a surface defined by an equation of the form: $w = |z|^2 + o(|z|^2)$, after a holomorphic change of coordinates, where $(z, w)$ are the coordinates of $\mathbb{C}^2$. The equivalence problem for real hypersurfaces defined by an equation of the form $\text{Im}(w) = |z|^2 + o((z, w)^2)$ was first studied by Poincaré at the beginning stage of the subject. A complete normal form, however, was only constructed in the work of Cartan-Chern-Moser much later. The problem of constructing the normal form for Bishop surfaces with a vanishing Bishop invariant has a very different nature from the hypersurface case, and had puzzled many mathematicians for more than two decades. Roughly speaking, constructing a normal form for hypersurfaces is equivalent to solving a certain infinite decoupled system of linear equations; while for the Moser problem, one needs to solve a very complicated un-decoupled infinite system of non-linear equations. Hence, the second Moser problem more or less asks to develop a non-linear version of the Chern-Moser theory in the hypersurface setting.

In my very recent long joint paper with Yin [1] published in Inventiones, we were finally able to give a solution to the equivalence problem for Bishop surfaces with a vanishing Bishop invariant. As a consequence, we answered, in the negative, the second problem that Moser asked in 1985, by showing that there are infinitely many holomorphic invariants in this setting. This is done in two major steps: We first derive the formal normal form for such surfaces. We then show that two real analytic Bishop surfaces with a vanishing Bishop invariant are holomorphically equivalent if and only if they have the same formal normal form (up to a trivial rotation). Our normal form was constructed by a completely new weighting system from what is used in the literature. This new new tool provides a very powerful way to solve a non-linear un-decoupled normalization system. Our convergence proof was done through a new hyperbolic geometry associated with the surface which has never been used in the literature before. We mentioned that the second problem of Moser is indeed much harder than the first one that was solved much earlier in [HK]. The work along these lines is the result of my tremendous efforts since 1995, and my more recent joint hard work with my former student Yin.

In [3], we considered the normal form problem for a real codimension two submanifold $M$
in $\mathbb{C}^n$ ($n > 2$) with an elliptic CR singularity at $p \in M$, where all generalized Bishop invariants are zero. We derived a pseudo-normal form for $M$ near $p$. As expected, the holomorphic structure of $M$ near $p$ is influenced not only by the nature of the CR singularity, but also by the fact that $(M, p)$ partially inherits the property of strongly pseudoconvex CR structures. This result is the first one linking together different types of the normal form theory such as the Chern-Moser normal form and the Moser pseudo-normal form. We also showed in [3], by a Nash-Moser type iteration procedure, that if all higher order invariants in the pseudonormal form obtained in [3] vanish, then $M$ is biholomorphically equivalent to the model. This result serves as the first step for the construction of the modular space for such objects, and is an important step for our current study on the regularity problem for complex Plateau problems of bounding a Levi-flat hypersurface by a real codimensional two submanifold in $\mathbb{C}^n$ with $n \geq 3$, which I describe below:

A natural geometric global problem associated with an even dimensional compact Bishop manifold $M$ in $\mathbb{C}^n$ is to ask when it bounds a smooth Levi-flat manifold $\tilde{M}$ with $M$ as its smooth boundary, as initiated in the work of Bishop. In the case of Bishop surfaces, there has been much work done along these lines by using the attached holomorphic disks. The existence and uniqueness for $\tilde{M}$ under certain natural conditions have been achieved by Bedford-Gaveau and others. The interior regularity for $\tilde{M}$ follows from the implicit function theorem in Banach spaces, while the boundary regularity (especially near the CR singular points) is fundamentally more complicated and has been accomplished in the work of Kenig-Webster, Moser-Webster and Huang-Krantz [HK].

As in the surface case, one is led to study the problem of bounding a Levi-flat hypersurface $\tilde{M}$ by a $2(n-1)$-manifold in $\mathbb{C}^n$ with exactly two elliptic CR singular points ($n \geq 3$). There is an obvious necessary condition for the problem to be solvable: $M$ should be non-minimal at every CR point, namely, for any CR point $p \in M$, there exists a CR submanifold in $M$ of CR dimension $(n-2)$ passing through $p$. Under this condition, Dolbeault-Tomassini-Zaitsev proved that $M$ always bounds a Levi-flat object $\tilde{M}$ but in a very weak sense of currents (corresponding to the weak solution in the PDE setting). However, the boundary regularity, as well as the interior regularity for $M$ remains open. This problem is the main concern in my joint paper in progress with Yin in [4]. The problem can also be formulated as the regularity problem of a fully non-linear degenerate elliptic equation, called the Levi-equation. One of the basic tools in this work will be the deformation theory of compact strongly pseudoconvex manifolds. The other main step for our investigation is a very careful study of the precise holomorphic structure near elliptic CR singular points as encountered in [3]. We expect to complete the project in [3] in the near future.

Another future problem which Yin and I would like to study is to construct the modular space for hyperbolic Bishop surfaces, which is related to the Ecalle and Voronin theory.

(2). Rigidity problems in Several Complex Variables: The study of rigidity problems of holomorphic mappings between complex manifolds has been one of the extensively studied subjects in complex analysis and geometry. The study of rigidity problems of holomorphic
mappings between complex manifolds reveals not just the beauty of the field itself, but also plays an important role in the application of complex analysis to the understanding of many problems in classical dynamics, differential geometry, Lie group theory, arithmetical algebraic geometry and others. Generally speaking, local holomorphic mappings are completely determined by their rank. Hence, rigidity problems of maps between complex manifolds usually can only be attacked by using the global invariants. For mappings between open manifolds with regular boundaries, it is often more convenient to reduce the study to that of the induced CR mappings between their boundaries, whose existence, even locally, requires certain matching-up conditions for CR invariants. In case the manifolds have good symmetry, one can derive, from the local matching-up equations for CR invariants, the globally defined partial differential equations. This then makes the hard analysis more usable to deal with these problems. This idea was first used in [Hu2] for solving a longstanding open question on a linearity problem for proper holomorphic mappings between balls with only $C^2$-smooth boundary regularity, initiated by the works of Poincaré, Alexander, Webster, Faran, Forstneric, D’Angelo, etc. In [Hu4], we found a new semi-linearity phenomenon for holomorphic maps between balls. Namely, we clarified exactly how the rigidities slowly disappear as the codimension increases. We showed that the phenomenon has the exact analogy with the classical invariant theory of Cartan-Janet concerning the obstructions for locally isometrically embedding analytic Riemannian manifolds into real Euclidean Spaces. The basic invariant theory used in [Hu2] and [Hu4] is more along the lines of Moser normal form theory. In [EHZ], we studied the rigidity property for CR submanifolds embedded in the sphere by using the more geometric Cartan-Chern-Moser-Webster theory.

The connections and applications of the work in [Hu2-4] [EHZ] to many other problems in invariant theory were further studied in our several other papers. There is a longstanding and notoriously hard open problem in Several Complex Variables which asks if there is a universal number $t$ such that any proper holomorphic map from the unit $n$-ball $B^n$ into the unit $N$-ball $B^N$ that is $C^t$-smooth up to the boundary must be rational ($n > 1$). As a continuation of the work in [Hu4], in [5], we were able to prove that any proper holomorphic map from $B^n$ into $B^N$ with $N \leq n(n + 1)/2$, that is three times differentiable up to the boundary, must be rational. This result strongly depends on the core theory developed in [Hu4] and still remains to be the best result along these lines. In [6], we found a new gap phenomenon (the second gap) for proper holomorphic maps between ball. The work in [EHZ] has an immediate application to the study of the complex structure of isolated normal singularities. This is because, all CR links are CR submanifolds embedded into the spheres. For instance, all Milnor exotic spheres carry out CR structures which can be realized as CR submanifolds of the standard spheres. Among other applications in [EHZ], we derived several strong super-rigidity properties for the complex structures of isolated complex singularities.

In [7], we proved the following rigidity theorem for complex singularities: Any isolated normal singularity whose CR link is an algebraic spherical CR manifold must be isomorphic to a quotient singularity. The result is obviously false when the link is just a real analytic spherical CR manifold, for the Grauert tube of a compact hyperbolic Riemann surface equipped
with a canonical metric is a real analytic spherical CR manifold which cannot bound a quotient singularity for its fundamental group is infinite. This result provides another evidence that complexity of the CR links is related to the complexity of the complex structure of the singularities in a very subtle way.

At this point, we should mention that the study for mappings between balls have attracted much attention in the past two decades. There have been important works done by Stensones, Fortsneric, D’Angelo, D’Angelo and his coauthors, and many others for different but related problems.

We have recently invested a lot of time on rigidity problems for holomorphic maps between Levi non-degenerate hypersurfaces with positive signature. In this setting, the mappings are even more rigid. In [8], we found a new admissible space for the Chern-Moser operator, that has many applications. Namely, we obtained a normal form for a class of Levi non-degenerate hypersurface different from that of Chern-Moser. The new normal form is more suitable in the investigation of various rigidity problems. In [9], a joint work with Baouendi, we first studied the super-rigidity problem for holomorphic mappings between hyper-quadrics with positive signature of any codimension. We proved in [9] that any holomorphic embedding between hyperquadrics with the same positive signature must be a totally geodesic embedding. This phenomenon is similar to that for holomorphic embeddings between bounded symmetric domains of rank at least two, as discovered by Siu, Mok and others. Recently, Ng has written up a paper in which connections with problems in Algebraic Geometry have been revealed. In [10], the Bonnet type of such a result has been established for mappings into hyperquadrics with the same positive signature, by using the more geometric invariant theory. In [11], we even proved a local version of the generalized rigidity result of [9], when the signature is different. In the global setting, as observed by Mok, the result in [11], together with the classical result of Feder for mappings from \( \mathbb{P}^n \) into \( \mathbb{P}^{2n-1} \), shows that any proper holomorphic map from the generalized ball with signature \( \ell \) in \( \mathbb{P}^n \) into the generalized ball with signature \( \ell' \) in \( \mathbb{P}^N \) must be linear if \( 0 < \ell' < 2\ell \). Here \( \ell \leq (n - 1)/2 \).

Along the lines of rigidities for holomorphic objects, we also studied in [HJ2] [HJ3] [HJY] the Riemann mapping problems for algebraic domains. We proved in [HJ2] that any bounded algebraic strongly pseudoconvex domain, which is locally spherical, must be biholomorphic to the unit ball. Notice that there are bounded locally spherical real analytic strongly pseudoconvex domains, that are not even topologically equivalent to the balls. We gave in [HJ3] an effective bound for the total Betti numbers for the automorphism groups of strongly pseudoconvex bounded algebraic domains in terms of the degree of the defining functions and the complex dimensions of the domains. We established in [Hu5] a type of the real version of the Chow’s theorem. We mention that the result in [Hu5] has been recently applied by Mok-Ng as a tool through the Grauert tube trick suggested by Siu to solve the Clozel-Ullmo conjecture from Arithmetic Number Theory.

There are many important open problems which can guide one to work further along these lines. (See, for instance, my survey paper with Ji [14]). Here we mention the rigidity problem for the embedding of compact hyperbolic space forms into a hyperbolic space form; the general
(3). CR deformation of compact strongly pseudoconvex manifolds and applications: A CR family of compact strongly pseudoconvex CR manifolds is a triplet $(X, \pi, \Delta)$ with $X$ a strongly pseudoconvex CR manifold of hypersurface type, $\pi$ a proper CR submersion and $\Delta$ the unit disk in $\mathbb{C}$. As compact CR manifolds often come as the smooth boundaries of complex spaces with isolated singularities, the CR family is modeled by the typical example of the holomorphic deformation of the complex structure of isolated singularities. Indeed, a CR family of compact strongly pseudoconvex CR manifolds more or less always comes in this manner as demonstrated in [12]. Hence, the study of the CR deformation theory is immediately linked with the holomorphic deformation theory of isolated singularities. There are several natural questions for a CR family. The first one is the simultaneous embedding problem. By the Montet de Monel theorem, each fiber is embeddable into a complex Euclidean space. Now, for any embedding $f_0$ of the initial fiber, can it extend to be a smooth CR map defined over $X$? The answer to this question is important for the study of the simultaneous blowing-down problem of the exceptional sets and for the study of the flatness of a family of complex singularities. There is a theorem by Tanaka in the smooth category under the Kodaira-Spence stability condition for the first Kohn-Rossi cohomology group. Tanaka’s extension theorem is to study the dependence of the $\Box_b$ operator of each fiber and can not be extended to the CR dependence (or holomorphic dependence) on the parameter. The second problem is associated with the fiber normality of the filling of $X$. By the work of Siu-Ling, $X$ admits a unique Stein normal filling $\hat{X}$, which automatically induces a holomorphic filling for each fiber $X_t$. It had been an open question for quite a long time when the filling resulting from $\hat{X}$ for each fiber is also a normal filling. Fujiki, in the early 1980’s, showed that this is the case under some depth condition for the singularity. His method is very algebraic and is also hard to extend to the general situation where the Kodaira-Spence stability condition holds. This problem is again related to the smooth extension problem of CR functions defined over $X_0$ to $X$. The extension problem can be formulated in terms of $\bar{\partial}$-equation. However, the $\bar{\partial}$-equation has to be solved over a complex manifold with quite singular boundary. The extension problem is closely related to the study of the Grauert direct image theorem for a family of $(1,1)$-convex-concave manifold. In [12], we proved a general smooth CR extension theorem for CR functions from $X_0$ to $X$, by solving $\bar{\partial}$-equation over a lunar manifold with mixed boundary condition through the method of Catlin and by applying the work of Siu-Ling on the generalization of the Grauert direct image theorem for $(1,1)$ convex-concave manifold. This, in particular, gives the simultaneous embedding theorem for the family under the Kodaira-Spencer stability condition for the first Kohn-Rossi cohomology group. This also answers the normality problem under the same condition, which is more or less the best possible result one can expect. Notice that earlier, algebraic geometers could only obtain, using the algebraic method, the result in the special case where the first Kohn-Rossi cohomology group is zero.

As a continuation of the work in [12], we will study the case when each fiber is three dimensional. Also, we would like to establish Kohn’s theory for a holomorphic family of $(1,1)$
convex-concave complex manifolds, and thus provide a new understanding of Siu-Ling’s work on the Grauert direct image theorem for such a family.

Remark that several other developments along these lines were also discussed in [7].

4. Chern-Moser-Weyl tensors and an embedding problem: There is a well known problem in Several Complex Variables that asks when a Levi non-degenerate hypersurface $M_\ell$ in $\mathbb{C}^{n+1}$ of signature $\ell$ with $0 \leq \ell \leq n/2$ can be embedded into a hyperquadric $H^{N+1}_\ell$ in $\mathbb{C}^{N+1}$ of the same signature for $N >> n$. By general invariant theory and a Baire category argument, Forstneric showed, twenty-five years ago, that most of such $M_\ell$'s are not smoothly (CR transversally) embedded into $H^{N+1}_\ell$ for any $l'$ and $N$. On the other hand, about 30 years ago, Webster showed that a Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ of signature $\ell$, defined by a real polynomial, can always be embedded into the hyperquadric $H^{\ell+1}_{\ell+1}$ of signature $\ell+1$ but in one dimension higher complex space. This has then led to an open problem to understand when an algebraic Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ can be embedded into a hyperquadric of the same signature but in a much higher complex space.

In [13], we gave a checkable necessary condition whether $M_\ell$ can be embedded into $H^{N+1}_\ell$ when $\ell \in (0, n/2)$. Our criterion is based on a monotonicity property for the Chern-Moser-Weyl tension along the cone defined by tangent vectors of type $(1,0)$ in the null space of the Levi-form. Roughly speaking, the monotonicity property that we get, says that a CR embedding from one Levi non-degenerate hypersurface into another one with the same signature decreases the Chern-Moser-Weyl curvature tensor. This phenomenon may be compared with other type of monotonicity properties for the sectional or bisectional curvatures under the application of holomorphic maps. It may find more applications in other studies. Since the hyperquadrics have the vanishing Chern-Moser-Weyl tensor, we were able to construct in [13] many simple algebraic Levi non-degenerate hypersurfaces which can never be embedded into a hyperquadric of the same signature. In the $\ell = 0$ case, our criterion does not work and it is still a nice problem to find a similar criterion as in [13] in this case. On the other hand, Zaitsev and the author have recently observed (in the Serra Negra SCV conference August, 2011, Brazil) that there are many simple examples of real algebraic strongly pseudoconvex hypersurfaces which can not be embedded into any closed strongly pseudoconvex hypersurface in any complex space (in particular, the Heisenberg hypersurfaces). An immediate example is $M := \{z \in \mathbb{C}^n : \Im z_n = |z|^2 - |z_1|^4\}$ with $n \geq 3$. If $M$ near 0 can be embedded into a closed algebraic strongly pseudoconvex hypersurface, then the embedding must be algebraic and thus extends to an embedding into a local embedding from a point with mixed signs of Levi form. This leads to a problem to find a compact real algebraic Levi non-degenerate hypersurface (may be sitting in the complex projective space) that is not locally embeddable into a hyperquadric with the same signature. This is one of the problems which we would like to look at carefully in the future.

In [15], the same method was further used together with a rescaling argument to study the CR transversality property for CR maps between Levi non-degenerate hypersurfaces with positive signature.

More recent works mentioned above
4. “Bounding a smooth Levi-flat hypersurface by a real codimensional two submanifold in \( \mathbb{C}^n \)” (with W. Yin), in a final stage of the preparation. (approximately 60 pages)
5. “On several properties for holomorphic maps from \( \mathbb{B}^n \) into \( \mathbb{B}^N \)” (with S. Ji and D. Xu), *Contemporary Math.*, a special issue in honor of F. Treves, 267-293, 2005.
11. “Holomorphic maps between hyperquadrics with small difference in signature” (with S. Baouendi and P. Ebenfelt), Accepted for publication in *American Journal of Mathematics*, 2011. (25 pages)
16. “Local holomorphic conformal maps between a class of Kähler manifolds” (with Yuan Yuan), 2011. (30 pages)

**Other earlier works mentioned above**


[HJ2]: “Global holomorphic extension of a local map and a Riemann mapping theorem for algebraic domains” (with S. Ji), Mathematical Research Letters 5, 247-260 (1998).


[HJY]: “An example of strongly pseudoconvex real analytic hypersurfaces which is not holomorphically equivalent to any algebraic hypersurfaces” (with S. Ji and S. S. T. Yau), Arkiv for Math 39, 2001, 75-93.