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Super-rigidity for CR embeddings of real hypersurfaces into hyperquadrics ☆

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Abstract

We consider holomorphic mappings sending a given Levi-nondegenerate pseudoconcave hypersurface M in $\mathbb{C}^{n+1}$ into a nondegenerate hyperquadric of the same signature in $\mathbb{PC}^{N+1}$ and show that if M is sufficiently close to a hyperquadric in a certain sense, then any two such mappings differ only by an automorphism of the hyperquadric.

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1. Introduction

It was discovered by Poincaré [22] that a local non-constant holomorphic mapping sending a piece of the unit sphere S in $\mathbb{C}^2$ into itself must in fact be a global holomorphic automorphism of $\mathbb{CP}^2$ (i.e. a projective linear map) preserving S. Almost fifty years later, Alexander [1] completed Poincaré's program along these lines in the equi-dimensional case, by showing that

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a non-constant holomorphic map sending an open piece of the unit sphere S in $\mathbb{C}^n$ into S for any $n \geqslant 2$ is also an automorphism of $\mathbb{CP}^n$ preserving S.

Webster [25] first obtained a similar rigidity result for holomorphic mappings (or sufficient smooth CR mappings) sending a piece of the unit sphere S^n in $\mathbb{C}^{n+1}$ into the unit sphere S^N in a different complex space $\mathbb{C}^{N+1}$ with $N = n + 1 \geqslant 3$. Cima and Suffridge in [5] conjectured that the just mentioned Poincaré–Alexander–Webster rigidity property holds for any C^2-smooth non-constant CR map, provided that the codimension $N - n < n$. This was verified by Faran in [11] when the map is real analytic. Forstneric's reflection principle in [12] (see also [6]) shows that it holds when the map is C^{N-n+1}-smooth. In [15], this rigidity was finally established for any non-constant C^2-smooth CR map. The bound $N < 2n$ is optimal as can be seen by examples such as the so-called Whitney map (see e.g. Example 1.1 in [8]). In fact, there are polynomial maps of arbitrarily high degree k sending the unit sphere $S^n \subset \mathbb{C}^{n+1}$ into a sphere $S^N \subset \mathbb{C}^{N+1}$ with N depending on n and k (see D'Angelo [7]). The reader is also referred to [10,16,13,17] for a classification of all rational maps in the case $n = 1$, $N = 2$, and in the case $N \leqslant 3n - 4$.

The situation is quite different in the case of maps between nondegenerate *pseudoconcave* hyperquadrics. An immediate benefit of the Lewy extension theorem (see Lewy [20]; see also Hörmander [14, Theorem 2.6.13]) in this consideration is that one needs only to deal with holomorphic maps instead of more general CR maps. More recently, it was shown in [3] that for such hyperquadrics there is no restriction on the codimension $N - n$ for the analogous rigidity phenomenon to hold. In the present paper, we study a more general situation where the source manifold is not necessarily a hyperquadric. We consider holomorphic mappings sending a given Levi-nondegenerate pseudoconcave hypersurface M in $\mathbb{C}^{n+1}$ into a nondegenerate hyperquadric of the same signature in $\mathbb{CP}^{N+1}$. We show that if M is sufficiently close to a hyperquadric in a certain sense (i.e. M can be embedded with low codimension in a hyperquadric of the same signature), then any two such mappings differ only by an automorphism of the hyperquadric (see Theorem 1.1 for the precise formulation). Previous results along these lines in the strictly pseudoconvex case include [25,8], and in the general Levi nondegenerate case [9]. The proof of our main result relies on the early work in the study of Pseudo-Hermitian geometry (see [24,25,19] and the references therein) and, in particular, the more recent derivations in [8] and [9].

Let $M \subset \mathbb{C}^{n+1}$ be a smooth hypersurface and $p \in M$. Assume that M is Levi nondegenerate at p and $\mathcal{L}: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ a representative of the Levi form of M at p. If we let e_- and e_+ be the number of negative and positive eigenvalues of $\mathcal{L}$, respectively, then $l(M, p) := \min(e_-, e_+) \leqslant n/2$ is independent of the choice of representative $\mathcal{L}$ of the Levi form. We shall refer to $l(M, p)$ as the signature of M at p. If M is connected and Levi nondegenerate at every point, then $l := l(M, p)$ is constant and we shall say that M has signature l.

We let $Q_l^N \subset \mathbb{CP}^{N+1}$ denote the standard hyperquadric of signature $0 \leqslant l \leqslant N/2$ given in homogeneous coordinates $[z_0 : z_1 : \ldots : z_{N+1}]$ by

$$-\sum_{j=0}^{l} |z_j|^2 + \sum_{k=l+1}^{N+1} |z_k|^2 = 0. \tag{1.1}$$

(Thus, the superscript in Q_l^N represents the CR dimension and the subscript represents the signature.) We observe that Q_l^N is a connected hypersurface of CR dimension N, which is Levi nondegenerate at every point. Its signature is l. We denote by $\mathrm{Aut}(Q_l^N)$ the subgroup of biholomorphic mappings of $\mathbb{CP}^{N+1}$ preserving Q_l^N. It is well known [4] that $\mathrm{Aut}(Q_l^N)$ can be identified with the group of invertible $(N+2) \times (N+2)$ matrices that preserve the quadratic form on the left-hand side of (1.1) (up to sign if $l = N/2$). We also note that if $2l \leqslant N_0 < N$, then the standard linear embedding $L : \mathbb{CP}^{N_0+1} \to \mathbb{CP}^{N+1}$, given by

$$L\big([z_0 : \ldots : z_{N_0+1}]\big) := [z_0 : \ldots : z_{N_0+1} : 0 : \ldots : 0], \tag{1.2}$$

satisfies $L(Q_l^{N_0}) \subset Q_l^N$.

To formulate our main result, we shall need one more definition. If $M \subset \mathbb{C}^{n+1}$ is a real hypersurface, then we shall say that M is *locally biholomorphically equivalent* to the hyperquadric Q_l^n at $p \in M$ if there are $p' \in Q_l^n$, open neighborhoods $U \subset \mathbb{C}^{n+1}$ and $V \subset \mathbb{CP}^{n+1}$ of p and p', respectively, and a biholomorphism $H : U \to V$ such that $H(M \cap U) = Q_l^n \cap V$ and $H(p) = p'$. Our main result is the following.

Theorem 1.1. *Let $M \subset \mathbb{C}^{n+1}$ be a connected real-analytic Levi-nondegenerate hypersurface of signature $l \leqslant n/2$. Moreover, if $l = n/2$, then assume that M is not locally biholomorphically equivalent to the hyperquadric $Q_{n/2}^n$ at any point of M. Suppose that there is an open connected neighborhood U of M in $\mathbb{C}^{n+1}$ and a holomorphic mapping $f_0 : U \to \mathbb{CP}^{N_0+1}$ with $f_0(M) \subset Q_l^{N_0}$ such that $f_0(U) \not\subset Q_l^{N_0}$. If $N \geqslant N_0$ and $f : U \to \mathbb{CP}^{N+1}$ is a holomorphic mapping with $f(M) \subset Q_l^N$, $f(U) \not\subset Q_l^N$, and $N_0 - n < l$, then there is $T \in \mathrm{Aut}(Q_l^N)$ such that $f := T \circ L \circ f_0$, where L denotes the standard linear embedding given by* (1.2).

The conclusion of Theorem 1.1 with the additional assumption that M is the hyperquadric Q_l^n (and $N_0 = n$, $f_0(z) \equiv z$) is contained in Theorem 1.6(i) of [3]. If the condition $N_0 - n < l$ is replaced by $N_0 + N < 3n$, then the conclusion of Theorem 1.1 follows from the work [8] (in the strictly pseudoconvex case $l = 0$) and [9] (in the general case). We conclude the introduction with a number of remarks.

Remark 1.2. We point out that if $M \subset \mathbb{C}^{n+1}$ is a merely smooth (C^∞) connected Levi-nondegenerate hypersurface of signature $l > 0$ and $F : M \to Q_l^N \subset \mathbb{CP}^{N+1}$ a smooth CR mapping, then F is the restriction to M of a holomorphic mapping $f : U \to \mathbb{CP}^{N+1}$, where U is an open neighborhood of M in $\mathbb{C}^{n+1}$. Indeed, this follows essentially from a classical result of Lewy [20] (see also Theorem 2.6.13 in [14]), since the Levi form of M has eigenvalues of both signs at every point. If, in addition, $f(U)$ is not contained in Q_l^N, then M is real-analytic. To see this, let p_0 be a point on M and $\rho = 0$ a real-analytic defining equation for Q_l^N (in some local chart) near $f(p_0)$. It follows that M is contained, near p_0, in the real-analytic variety V defined by $\rho \circ f = 0$. Since $f(U) \not\subset Q_l^N$, it follows that $\rho \circ f \not\equiv 0$ and hence V is non-trivial. The real-analyticy of M now follows from a theorem of Malgrange [21]. Hence, the conditions in Theorem 1.1 that M is real-analytic and f_0, f are holomorphic can be weakened to M being smooth and f_0, f being CR with the appropriate conditions on their holomorphic extensions.

Remark 1.3. We also remark that if $M \subset \mathbb{C}^{n+1}$ is a connected real-analytic Levi-nondegenerate hypersurface of signature l and M is locally biholomorphically equivalent to the hyperquadric Q_l^n at some point $p \in M$, then M is locally biholomorphically equivalent to Q_l^n at every point in M. Indeed, this follows from the fact that M is locally biholomorphically equivalent to Q_l^n at p if and only if the CR curvature of M (see below) vanishes identically in an open neighborhood of p in M. The conclusion above now follows from the real-analyticy of the CR curvature of M and the connectedness of M. Hence, the additional assumption in Theorem 1.1 when $l = n/2$

that M is not locally biholomorphically equivalent to $Q^n_{n/2}$ at any point of M can be replaced by the seemingly weaker condition that M is not locally biholomorphically equivalent to $Q^n_{n/2}$ at one point in M.

Remark 1.4. If M is locally biholomorphically equivalent to $Q^n_{n/2}$ at some point $p \in M$ (and hence at every point of M by Remark 1.3), then the conclusion of Theorem 1.1 does not hold in general. However, the situation can be reduced to one considered in [3] as follows. Under the assumption above, we may take $N_0 = n$ in the statement of Theorem 1.1 and, by shrinking U if necessary, we may assume that $f_0 : U \to \mathbb{CP}^{n+1}$ is a biholomorphism (onto its image) sending M into $Q^n_{n/2}$. Let f be as in the statement of Theorem 1.1. By applying Theorem 1.6 in [3] to the mapping $f \circ f_0^{-1}$, we conclude that $f = T \circ L \circ T_0 \circ f_0$, where T and L are as in Theorem 1.1 and T_0 is either the identity in $\mathbb{CP}^{n+1}$ or the flip

$$[z_0 : z_1 : \ldots : z_n : z_{n+1}] \mapsto [z_{n+1} : z_n : \ldots : z_1 : z_0]. \tag{1.3}$$

We note that it is not always possible to take T_0 to be the identity in this situation.

Remark 1.5. If there is an open connected neighborhood U of M in $\mathbb{C}^{n+1}$ and a holomorphic mapping $f_0 : U \to \mathbb{CP}^{N_0+1}$ with $f_0(M) \subset Q^{N_0}_l$ such that $f_0(U) \not\subset Q^{N_0}_l$, then necessarily $N_0 \geqslant n$. Indeed, if $N_0 < n$, then the rank of f_0 would be $\leqslant n$ at every point of M. Theorem 5.1 in [2] would then imply that $f_0(U) \subset Q^{N_0}_l$ contradicting the hypothesis above.

2. Two basic lemmas

In this section, we shall formulate two lemmas that are key ingredients in the proof of Theorem 1.1. The first lemma was proved in [15] and [9], [15, Lemma 3.2]. For the reader's convenience, we reproduce its statement here.

Lemma 2.1. *Let k, l, n be nonnegative integers such that $1 \leqslant k < n$. Assume that $g_1, \ldots, g_k$, $f_1, \ldots, f_k$ are germs at $0 \in \mathbb{C}^n$ of holomorphic functions such that*

$$\sum_{i=1}^{k} g_i(z)\overline{f_i(z)} = A(z, \bar{z})\left(-\sum_{i=1}^{l} |z_i|^2 + \sum_{j=l+1}^{n} |z_j|^2\right), \tag{2.1}$$

where $A(z, \zeta)$ is a germ at $0 \in \mathbb{C}^n \times \mathbb{C}^n$ of a holomorphic function. Then $A(z, \bar{z}) \equiv 0$.

In [15], Lemma 2.1 is stated only for $l = 0$, but the proof for $l > 0$ is identical (see Lemma 3.1 in [9]). Lemma 2.1 was also a crucial tool in the papers [15,8,9]. The second lemma that we shall need is the following.

Lemma 2.2. *Let k, l, n be nonnegative integers such that $k < l \leqslant n/2$. Assume that $g_1, \ldots, g_k, f_1, \ldots, f_m$ are germs at $0 \in \mathbb{C}^n$ of holomorphic functions such that*

$$-\sum_{i=1}^{k} |g_i(z)|^2 + \sum_{j=1}^{m} |f_j(z)|^2 = A(z, \bar{z})\left(-\sum_{i=1}^{l} |z_i|^2 + \sum_{j=l+1}^{n} |z_j|^2\right), \tag{2.2}$$

where $A(z, \zeta)$ is a germ at $0 \in \mathbb{C}^n \times \mathbb{C}^n$ of a holomorphic function. Then $A(z, \bar{z}) \equiv 0$.

The proof of Lemma 2.2 can be found in [3, Lemma 4.1] (with $\ell' = \ell$ and after a direct application of Lemma 2.1 of [3]). The lemma also follows in a straightforward way from Theorem 5.7 in the subsequent work [2].

3. Preliminaries

We shall use the set-up and notation of [8]. The reader is referred to that paper for the terminology used below and a brief introduction to the pseudohermitian geometry and the CR pseudoconformal geometry. (The reader is of course also referred to the original papers by Chern and Moser [4], Webster [24], and Tanaka [23].) Although the main focus of [8] is on strictly pseudoconvex hypersurfaces, many of the results obtained in that paper work equally well for Levi-nondegenerate hypersurfaces and we shall use those results in this paper. Thus, let M be a Levi-nondegenerate CR-manifold of dimension $2n+1$, with rank n CR bundle $\mathcal{V}$, and signature $l \leqslant n/2$. Near a distinguished point $p_0 \in M$, we let θ be a contact form and T its characteristic (or Reeb) vector field, i.e. the unique real vector field that satisfies

$$T \lrcorner d\theta = 0, \qquad \langle \theta, T \rangle = 1.$$

We complete θ to an admissible coframe $(\theta, \theta^1, \ldots, \theta^n)$ for the bundle $T'M$ of $(1,0)$-cotangent vectors (i.e. the cotangent vectors that annihilate $\mathcal{V}$). Recall that the coframe is called admissible if $\langle \theta^\alpha, T \rangle = 0$, for $\alpha = 1, \ldots, n$. We choose a frame $L_1, \ldots, L_n$ for the bundle $\bar{\mathcal{V}}$, or, as we shall also refer to it, the bundle of $(1,0)$-tangent vectors $T^{1,0}M$. The frame for $T^{1,0}M$ will be chosen such that $(T, L_1, \ldots, L_n, L_{\bar{1}}, \ldots, L_{\bar{n}})$ is a frame for $\mathbb{C}TM$, near p_0, which is dual to the coframe $(\theta, \theta^1, \ldots, \theta^n, \theta^{\bar{1}}, \ldots, \theta^{\bar{n}})$. Here and in what follows, $L_{\bar{\alpha}} = \overline{L_\alpha}$, $\theta^{\bar{\alpha}} = \overline{\theta^\alpha}$, etc. We shall denote the matrix representing the Levi form (relative to the frame $L_1, \ldots, L_n$) by $(g_{\alpha\bar{\beta}})$, where $\alpha, \beta = 1, \ldots, n$. We may assume that $g_{\alpha\bar{\beta}}$ is constant, in fact that it is diagonal with diagonal elements $-1, \ldots, -1$ (l times) and $1, \ldots, 1$ ($n-l$ times), although this fact will not be explicitly used most of the time. We denote by ∇ the Webster–Tanaka pseudohermitian connection on $\bar{\mathcal{V}}$, which is expressed relative to the chosen frame and coframe by

$$\nabla L_\alpha := \omega_\alpha{}^\beta \otimes L_\beta, \tag{3.1}$$

where the 1-forms $\omega_\alpha{}^\beta$ on M are uniquely determined by the conditions

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha{}^\beta \mod \theta \wedge \theta^{\bar{\alpha}}, \\ dg_{\alpha\bar{\beta}} &= \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha}. \end{aligned} \tag{3.2}$$

Here and for the remainder of this paper, we use the summation convention that an index that appears both as a subscript and superscript is summed over. We also use the Levi form to raise and lower indices in the usual way. The first condition in (3.2) can be rewritten as

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta + \theta \wedge \tau^\beta, \quad \tau^\beta = A^\beta{}_{\bar{\nu}} \theta^{\bar{\nu}}, \quad A^{\alpha\beta} = A^{\beta\alpha} \tag{3.3}$$

for a suitable uniquely determined torsion matrix $(A^{\beta}{}_{\bar{\alpha}})$, where the last symmetry relation holds automatically (see [24]). For future reference, we record here also the fact that the coframe $(\theta, \theta^1, \dots, \theta^n)$ is admissible if and only if

$$d\theta = i g_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \theta^{\bar{\beta}}. \tag{3.4}$$

Now, let $\hat{M}$ be a Levi-nondegenerate CR-manifold of dimension $2\hat{n}+1$, with rank $\hat{n}$ CR bundle $\hat{\mathcal{V}}$ $(= \overline{T^{1,0}\hat{M}})$, and signature $\hat{l} \leqslant \hat{n}/2$. Let $f : M \to \hat{M}$ be a smooth CR mapping. Our arguments in the sequel will be of a local nature and we shall restrict our attention to a small open neighborhood of p_0 (that we still shall refer to as M). We shall use a $\hat{}$ to denote objects associated to $\hat{M}$. Capital Latin indices A, B, etc., will run over the set $\{1, 2, \dots, \hat{n}\}$ whereas Greek indices α, β, etc., will run over $\{1, 2, \dots, n\}$ as above. Moreover, we shall let small Latin indices a, b, etc., run over the complementary set $\{n+1, n+2, \dots, \hat{n}\}$. Recall that $f : M \to \hat{M}$ is a CR mapping if

$$f^*(\hat{\theta}) = a\theta, \qquad f^*(\hat{\theta}^A) = E^A{}_{\alpha}\theta^{\alpha} + E^A\theta, \tag{3.5}$$

where a is a real-valued function and $E^A{}_{\alpha}$, E^A are complex-valued functions defined near p_0. We shall assume that f is *CR transversal* to $\hat{M}$ at $p_0 \in M$, which in the present context can be expressed by saying that $a(p_0) \neq 0$, where a is the function in (3.5). Without loss of generality, we may assume that $a \equiv 1$ (i.e. we take $\theta = f^*(\hat{\theta})$ in our admissible coframe $(\theta, \theta^{\alpha})$). We note that the CR transversality of f implies that $n \leqslant \hat{n}$. Indeed, it follows easily from (3.4) and (3.5) that

$$g_{\alpha\bar{\beta}} = \hat{g}_{A\bar{B}} E^A_{\alpha} E^{\bar{B}}_{\bar{\beta}}. \tag{3.6}$$

Since the rank of the matrices $(g_{\alpha\bar{\beta}})$ and $(\hat{g}_{A\bar{B}})$ are n and $\hat{n}$, respectively, we conclude that $n \leqslant \hat{n}$ and the rank of the matrix $(E^A{}_{\alpha})$ is n. Hence, if f is CR tranversal to $\hat{M}$, it also follows that f is an embedding, locally near p_0. We may assume, without loss of generality (by renumbering the $\hat{\theta}^A$ if necessary), that the admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ on $\hat{M}$ is such that the pullback $(\theta, \theta^{\alpha}) := (f^*(\hat{\theta}), f^*(\hat{\theta}^{\alpha}))$ is a coframe for M. Assume that $(\theta, \theta^{\alpha})$ defined in this way is also admissible. Hence, we shall drop the $\hat{}$ over the frames and coframes if there is no ambiguity. It will be clear from the context if a form is pulled back to M or not. Under the assumptions above, we shall identify M with the submanifold $f(M)$ of $\hat{M}$ and write $M \subset \hat{M}$. Then $T^{1,0}M$ becomes a rank n subbundle of $T^{1,0}\hat{M}$ along M. It follows that the (real) codimension of M in $\hat{M}$ is $2(\hat{n}-n)$ and that there is a rank $(\hat{n}-n)$ subbundle $N'M$ of $T'\hat{M}$ along M consisting of 1-forms on $\hat{M}$ whose pullbacks to M (under f) vanish. We shall call $N'M$ the *holomorphic conormal bundle of M* in $\hat{M}$. We shall say that the pseudohermitian structure $(\hat{M}, \hat{\theta})$ (or simply $\hat{\theta}$) is *admissible for the pair* $(M, \hat{M})$ if the characteristic vector field $\hat{T}$ of $\hat{\theta}$ is tangent to M (and hence its restriction to M coincides with the characteristic vector field T of θ). If the admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ on $\hat{M}$ is such that $(\theta, \theta^{\alpha})$, with $\theta := f^*(\hat{\theta})$, $\theta^{\alpha} := f^*(\hat{\theta}^{\alpha})$, is an admissible coframe on M and $f^*(\hat{\theta}^a) = 0$, then $(\hat{M}, \hat{\theta})$ is admissible for the pair $(M, \hat{M})$.

It is easily seen that not all contact forms $\hat{\theta}$ are admissible for $(M, \hat{M})$. However, Lemma 4.1 in [8] (which, though stated only for strictly pseudoconvex CR-manifolds, holds also for general Levi nondegenerate CR-manifolds) asserts that any contact form θ on M can be extended to a contact form $\hat{\theta}$ in a neighborhood of M in $\hat{M}$ such that $\hat{\theta}$ is admissible for $(M, \hat{M})$. Let us fix a contact form θ on M, extend it to an admissible contact form $\hat{\theta}$ for the pair $(M, \hat{M})$. We denote by $\hat{T}$ the characteristic vector field of $\hat{\theta}$ and by T its restriction to M. Recall that $T^{1,0}M$ is a rank n subbundle of the rank $\hat{n}$ bundle $T^{1,0}\hat{M}$. The Levi form of M at a point $p \in M \subset \hat{M}$ is given, under these identifications, by the restriction of the Levi form of $\hat{M}$ to the subspace $T_p^{1,0}M \subset T_p^{1,0}\hat{M}$ (and, hence, in particular, $\hat{l} \geqslant l$). If we let (L_α) be a frame for $T^{1,0}M$ such that the Levi form $g_{\alpha\bar{\beta}}$ of M is constant and diagonal with $-1, \ldots, -1$ (l times) and $1, \ldots, 1$ ($n-l$ times) on the diagonal, then we may complete (L_α) to a frame $(\hat{L}_A) = (L_\alpha, \hat{L}_a)$ for $T^{1,0}\hat{M}$ along M such that the Levi form $\hat{g}_{A\bar{B}}$ of $\hat{M}$ along M is constant and diagonal with diagonal elements $-1, \ldots, -1$ (l times), $1, \ldots, 1$ ($n-l$ times), $-1, \ldots, -1$ ($\hat{l}-l$ times) and $1, \ldots, 1$ ($\hat{n}-n-\hat{l}+l$ times). Finally, we extend the $\hat{L}_A$ to a neighborhood of M such that the Levi form of $\hat{M}$ stays constant. If we now let $(\hat{\theta}, \hat{\theta}^A, \hat{\theta}^{\bar{A}})$ be the dual coframe of $(\hat{T}, \hat{L}_A, \hat{L}_{\bar{A}})$, then clearly the coframe $(\hat{\theta}, \hat{\theta}^A)$ for $T'\hat{M}$ is admissible, its pullback to M equals $(\theta, \theta^\alpha, 0)$ and (θ, θ^α) is an admissible coframe for $T'M$. In other words, we have obtained the following result, in whose formulation we have taken a little more care to distinguish between M and its image $f(M)$ in $\hat{M}$. A similar result was obtained in [8] (Corollary 4.2) for strictly pseudoconvex hypersurfaces.

Proposition 3.1. *Let M and $\hat{M}$ be Levi-nondegenerate CR-manifolds of dimensions $2n+1$ and $2\hat{n}+1$, and signatures $l \leqslant n/2$ and $\hat{l} \leqslant \hat{n}/2$, respectively. Let $f : M \to \hat{M}$ be a CR mapping that is CR transversal to $\hat{M}$ along M. If (θ, θ^α) is any admissible coframe on M, then in a neighborhood of any point $\hat{p} \in f(M)$ in $\hat{M}$ there exists an admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ on $\hat{M}$ with $f^*(\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^a) = (\theta, \theta^\alpha, 0)$. In particular, $\hat{\theta}$ is admissible for the pair $(f(M), \hat{M})$, i.e. the characteristic vector field $\hat{T}$ is tangent to $f(M)$. If the Levi form of M with respect to (θ, θ^α) is constant and diagonal with $-1, \ldots, -1$ (l times) and $1, \ldots, 1$ ($n-l$ times) on the diagonal, then $(\hat{\theta}, \hat{\theta}^A)$ can be chosen such that the Levi form of $\hat{M}$ relative to this coframe is constant and diagonal with diagonal elements $-1, \ldots, -1$ (l times), $1, \ldots, 1$ ($n-l$ times), $-1, \ldots, -1$ ($\hat{l}-l$ times) and $1, \ldots, 1$ ($\hat{n}-n-\hat{l}+l$ times). With this additional property, the coframe $(\hat{\theta}, \hat{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n,l) \times U(\hat{n}-n, \hat{l}-l)$.*

Let us fix an admissible coframe (θ, θ^α) on M and let $(\hat{\theta}, \hat{\theta}^A)$ be an admissible coframe on $\hat{M}$ near a point $\hat{p} \in f(M)$. We shall say that $(\hat{\theta}, \hat{\theta}^A)$ is *adapted* to (θ, θ^α) on M (or simply to M if the coframe on M is understood) if it satisfies the conclusion of Proposition 3.1 with the requirement there for the Levi form. For convenience of notation though, we continue to denote the Levi forms by $g_{\alpha\bar{\beta}}$ and $\hat{g}_{A\bar{B}}$.

For ease of notation, we shall write (θ, θ^A) for the coframe $(\hat{\theta}, \hat{\theta}^A)$. The fact that (θ, θ^A) is adapted to M implies, in view of (3.3), that if the pseudohermitian connection matrix of $(\hat{M}, \hat{\theta})$ is $\hat{\omega}_B{}^A$, then that of (M, θ) is (the pullback of) $\hat{\omega}_\beta{}^\alpha$. Similarly, the pulled back torsion $\hat{\tau}^\alpha$ is τ^α. Hence omitting a ˆ over these pullbacks will not cause any ambiguity and we shall do it in the sequel. By our normalization of the Levi form, the second equation in (3.2) reduces to

$$\omega_{B\bar{A}} + \omega_{\bar{A}B} = 0, \tag{3.7}$$

where as before $\omega_{\bar{A}B} = \overline{\omega_{A\bar{B}}}$.

The matrix of 1-forms $(\omega_\alpha{}^b)$ pulled back to M defines the *second fundamental form* of M (or more precisely of the embedding f). Since θ^b is 0 on M, we deduce by using Eq. (3.3) that, on M,

$$\omega_\alpha{}^b \wedge \theta^\alpha + \tau^b \wedge \theta = 0, \tag{3.8}$$

which implies that

$$\omega_\alpha{}^b = \omega_\alpha{}^b{}_\beta \theta^\beta, \quad \omega_\alpha{}^b{}_\beta = \omega_\beta{}^b{}_\alpha, \quad \tau^b = 0. \tag{3.9}$$

As in [8], we identify the CR-normal space $T_p^{1,0}\hat{M}/T_p^{1,0}M$ with $\mathbb{C}^{\hat{n}-n}$ by letting the equivalence classes of the L_a form a basis in the former space. We consider the components of the second fundamental form $(\omega_\alpha{}^a{}_\beta)_{a=n+1,\dots,\hat{n}} = \omega_\alpha{}^a{}_\beta L_a$, for $\alpha, \beta = 1, \dots, n$, as vectors in the CR-normal space $\cong \mathbb{C}^{\hat{n}-n}$. We also view the second fundamental form $\omega_\alpha{}^a{}_\beta$ as a section over M of the vector bundle of $\mathbb{C}$-bilinear maps

$$T_p^{1,0}M \times T_p^{1,0}M \to T_p^{1,0}\hat{M}/T_p^{1,0}M, \quad p \in M.$$

For sections of this bundle we have the covariant differential induced by the pseudohermitian connections ∇ and $\hat{\nabla}$ on M and $\hat{M}$ respectively:

$$\nabla \omega_\alpha{}^a{}_\beta = d\omega_\alpha{}^a{}_\beta - \omega_\mu{}^a{}_\beta \omega_\alpha{}^\mu + \omega_\alpha{}^b{}_\beta \omega_b{}^a - \omega_\alpha{}^a{}_\mu \omega_\beta{}^\mu. \tag{3.10}$$

We use e.g. $\omega_\alpha{}^a{}_{\beta;\gamma}$ to denote its component in the direction θ^γ. Higher order covariant derivatives $\omega_\alpha{}^a{}_{\beta;\gamma_1,\dots,\gamma_l}$ are defined inductively in a similar way:

$$\nabla \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_j} = d\omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_j} + \omega_{\gamma_1}{}^b{}_{\gamma_2;\gamma_3\dots\gamma_j}\omega_b{}^a - \sum_{l=1}^{j} \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_{l-1}\mu\gamma_{l+1}\dots\gamma_j}\omega_{\gamma_l}{}^\mu. \tag{3.11}$$

As above, we also consider the covariant derivatives as vectors in $\mathbb{C}^{\hat{n}-n} \cong T_p^{1,0}\hat{M}/T_p^{1,0}M$ via the identification

$$\left(\omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_j}\right)_{a=n+1,\dots,\hat{n}} = \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_j} L_a.$$

We define an increasing sequence of vector spaces

$$E_2(p) \subset \dots \subset E_k(p) \subset \dots \subset \mathbb{C}^{\hat{n}-n} \cong T_p^{1,0}\hat{M}/T_p^{1,0}M$$

by letting $E_k(p)$ be the span of the vectors

$$\left(\omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\dots\gamma_j}\right)_{a=n+1,\dots,\hat{n}}, \quad \forall 2 \leqslant j \leqslant k,\ \gamma_i \in \{1, \dots, n\},$$

evaluated at $p \in M$. We shall say that the mapping $f : M \to \hat{M}$ is *constantly* (k,s)*-degenerate* at p (following Lamel [18], see [8]) if the vector space $E_k(q)$ has constant dimension $\hat{n} - n - s$ for q in an open neighborhood of p, $E_{k+1}(q) = E_k(q)$, and k is the smallest integer with this property.

4. The second fundamental form, covariant derivatives, and the Gauss equation

For the proof of our main results, we need to recall some further results and terminology from [8]. We keep the notation from the previous section. A tensor $T_{\alpha_1\ldots\alpha_r\bar\beta_1\ldots\bar\beta_s}{}^{a_1\ldots a_t\bar b_1\ldots\bar b_q}$, with $r,s\geqslant 1$, is called *conformally flat* if it is a linear combination of $g_{\alpha_i\bar\beta_j}$ for $i=1,\ldots,r$, $j=1,\ldots,s$, i.e.

$$T_{\alpha_1\ldots\alpha_r\bar\beta_1\ldots\bar\beta_s}{}^{a_1\ldots a_t\bar b_1\ldots\bar b_q}=\sum_{i=1}^{r}\sum_{j=1}^{s}g_{\alpha_i\bar\beta_j}(T_{ij})_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_r\bar\beta_1\ldots\widehat{\bar\beta_j}\ldots\ldots\bar\beta_s}{}^{a_1\ldots a_t\bar b_1\ldots\bar b_q},\tag{4.1}$$

where e.g. $\widehat{\alpha}$ means omission of that factor. (A similar definition can be made for tensors with different orderings of indices.) The following observation gives a motivation for this definition. Let $T_{\alpha_1\ldots\alpha_r\bar\beta_1\ldots\bar\beta_s}{}^{a_1\ldots a_t\bar b_1\ldots\bar b_q}$ be a tensor, symmetric in $\alpha_1,\ldots,\alpha_r$ as well as in $\beta_1,\ldots,\beta_s$, and form the homogeneous vector-valued polynomial of type (r,s) whose components are given by

$$T^{a_1\ldots a_t\bar b_1\ldots\bar b_q}(\zeta,\bar\zeta):=T_{\alpha_1\ldots\alpha_r\bar\beta_1\ldots\bar\beta_s}{}^{a_1\ldots a_t\bar b_1\ldots\bar b_q}\zeta^{\alpha_1}\ldots\zeta^{\alpha_r}\overline{\zeta^{\beta_1}}\ldots\overline{\zeta^{\beta_s}},$$

where $\zeta=(\zeta^1,\ldots,\zeta^n)$. Then, the reader can check that the tensor is conformally flat if and only if all the polynomials $T^{a_1\ldots a_t\bar b_1\ldots\bar b_q}(\zeta,\bar\zeta)$ are divisible by the Hermitian form $g(\zeta,\bar\zeta):=g_{\alpha\bar\beta}\zeta^\alpha\overline{\zeta^\beta}$. Since $\nabla g_{\alpha\bar\beta}=0$ (see the second equation of (3.2)), it is clear that covariant derivatives of a conformally flat tensor is again conformally flat.

We shall now restrict our attention to the case where the target manifold $\hat M$ is the standard hyperquadric Q_l^N in $\mathbb{CP}^{N+1}$, as defined by (1.1). (Thus, in what follows the CR dimension of $\hat M=Q_l^N$ is N.) The crucial property of the quadric that we shall use is that its Chern–Moser pseudoconformal curvature tensor $\hat S_{A\bar BC\bar D}$ vanishes identically. We shall need the following lemma. The corresponding result in the strictly pseudoconvex case is proved, but not explicitly stated in [8]. Although the proof in the general case is identical to that of the strictly pseudoconvex case, we give it here for the convenience of reader.

Lemma 4.1. *Let $M\subset\mathbb{C}^{n+1}$ be a smooth Levi-nondegenerate hypersurface of signature $l\leqslant n/2$, $f:M\to Q_l^N\subset\mathbb{C}^{N+1}$ a smooth CR mapping that is CR transversal to Q_l^N along M, and $\omega_\alpha{}^a{}_\beta$ its second fundamental form. Then, the covariant derivative tensor $\omega_\alpha{}^a{}_{\beta;\bar\gamma}$ is conformally flat.*

Proof. We shall work locally near a point $p\in M$ and use the setup introduced in Section 3. Let

$$\big(\omega,\omega^\alpha,\omega^{\bar\alpha},\phi,\phi_\beta{}^\alpha,\phi^\alpha,\phi^{\bar\alpha},\psi\big),\qquad\big(\hat\omega,\hat\omega^A,\hat\omega^{\bar A},\hat\phi,\hat\phi_B{}^A,\hat\phi^A,\hat\phi^{\bar A},\hat\psi\big)\tag{4.2}$$

be the Chern–Moser pseudoconformal connections on the coframe bundles $Y\to M$ and $\hat Y\to Q_l^N$, respectively, pulled back to M and Q_l^N by (the completion of) our admissible coframes $(\theta,\theta^\alpha,\theta^{\bar\alpha})$ and $(\theta,\theta^A,\theta^{\bar A})$ (see [8, Section 3]). The latter connection is then pulled back to M by the embedding f. The 1-form $\hat\phi_\alpha{}^a$ is of the form

$$\hat\phi_\alpha{}^a=\omega_\alpha{}^a{}_\beta\theta^\beta+\hat D_\alpha{}^a\theta,\tag{4.3}$$

for some coefficients $\hat D_\alpha{}^a$ (see (3.3), (3.6) of [24], or Proposition 3.1 in [8]). By differentiating (4.3), using the structure equation for $\hat\phi_\alpha{}^a$ ((3.12) in [8]; recall that the pseudoconformal curvature $\hat{S}_{A\bar{B}C\bar{D}}$ of Q_l^N vanishes identically), and identifying the coefficients of $\theta^\beta \wedge \theta^{\bar{\gamma}}$, we obtain

$$\omega_\alpha{}^a{}_{\beta;\bar{\gamma}} = i\big(g_{\alpha\bar{\gamma}}\hat{D}_\beta{}^a + g_{\beta\bar{\gamma}}\hat{D}_\alpha{}^a\big), \tag{4.4}$$

which proves the lemma. Here, to simplify the computation, we choose an adapted coframe near p, the point under study, such that $\omega_\alpha{}^\beta(p) = \hat{\omega}_a{}^b(p) = 0$ (cf. e.g. Lemma 2.1 in [19]). We will do the same in the following lemma, too. $\square$

We shall also need the following result that describes how covariant derivatives commute. A similar result (with a slightly stronger conclusion) can be found in [8] (Lemma 7.4). The proof given there uses a result that does not immediately apply to our current situation. We give therefore a (more or less) self-contained proof here.

Lemma 4.2. *Let M, f, and $\omega_\alpha{}^a{}_\beta$ be as in Lemma* 4.1. *Then, for any $s \geqslant 2$, we have a relation*

$$\omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\gamma_s\alpha\bar{\beta}} - \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\gamma_s\bar{\beta}\alpha} = C^a{}_{\gamma_1\ldots\gamma_s\alpha\bar{\beta}}{}^{\mu_1\ldots\mu_s}{}_b\,\omega_{\mu_1}{}^b{}_{\mu_2;\mu_3\ldots\mu_s} + T_{\gamma_1\ldots\gamma_s\alpha\bar{\beta}}{}^a, \tag{4.5}$$

where the tensor $C^a{}_{\gamma_1\ldots\gamma_s\alpha\bar{\beta}}{}^{\mu_1\ldots\mu_s}{}_b$ depends only on (θ,θ^α) and the second fundamental form $\omega_\alpha{}^a{}_\beta$, and $T_{\gamma_1\ldots\gamma_s\alpha\bar{\beta}}{}^a$ is conformally flat.

Proof. We shall use the pseudoconformal connections in (4.2), as in the proof of Lemma 4.1 above. By observing that the left-hand side of the identity (4.5) is a tensor, it is enough to show, for each fixed point $p \in M$, the identity at p with respect to any particular choice of adapted coframe (θ,θ^A) near p. By making a suitable unitary change of coframe $\theta^\alpha \to u_\beta{}^\alpha\theta^\beta$ and $\theta^a \to u_b{}^a\theta^b$ (in the tangential and normal directions respectively), we may choose an adapted coframe near p such that $\omega_\alpha{}^\beta(p) = \hat{\omega}_a{}^b(p) = 0$ (cf. e.g. Lemma 2.1 in [19]). By using (3.11) and (4.3), we conclude that, relative to this coframe, the left-hand side of (4.5) evaluated at p is equal to, modulo a conformally flat tensor, the coefficient in front of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ in the expression

$$\sum_{j=1}^{s} \omega_{\gamma_1}{}^a{}_{\gamma_2;\gamma_3\ldots\gamma_{j-1}\mu\gamma_{j+1}\gamma_s}\,d\omega_{\gamma_j}{}^\mu - \omega_{\gamma_1}{}^b{}_{\gamma_2;\gamma_3\ldots\gamma_s}\,d\hat{\phi}_b{}^a. \tag{4.6}$$

The first term (i.e. the sum) in (4.6) is clearly of the form on the right-hand side of (4.5). Indeed, the coefficients $d\omega_{\gamma_j}{}^\mu$ corresponding to the $C^a{}_{\gamma_1\ldots\gamma_s\alpha\bar{\beta}}{}^{\mu_1\ldots\mu_s}{}_b$ on the right in (4.5) only depend on the coframe (θ,θ^α) (and not even on the second fundamental form). It is not clear that the corresponding coefficients $d\hat{\phi}_b{}^a$ in the second term of (4.6) depend only on the coframe and the second fundamental form. To show that it does, we compute $d\hat{\phi}_b{}^a$ using the structure equation (3.12) in [8], the vanishing of θ^a on M, and the vanishing of $\hat{\phi}_\beta{}^\alpha$ and $\hat{\phi}_b{}^a$ at p modulo θ to obtain:

$$d\hat{\phi}_b{}^a = \hat{\phi}_b{}^\mu \wedge \hat{\phi}_\mu{}^a - i\delta_b{}^a\hat{\phi}_\mu \wedge \theta^\mu \quad \bmod \theta. \tag{4.7}$$

Making use of the fact that $\hat{\phi}_b{}^\mu = -\overline{\hat{\phi}_\mu^b}$ mod (θ), we see that the first term on the right-hand side of (4.7) contributes the term

$$g^{\mu\bar{\kappa}}g_{b\bar{c}}\,\omega_\mu{}^a{}_\gamma\,\omega_{\bar{\kappa}}{}^{\bar{c}}{}_{\bar{\nu}},$$

to the coefficient in front of $\theta^\alpha \wedge \theta^{\bar\beta}$ in (4.6). We observe that these only depend on the coframe and the second fundamental form. For the second term on the right in (4.7), we recall from [8] (see Eqs. (6.1) and (6.8)) that, pulled back to M,

$$\hat\phi^\alpha = \phi^\alpha + C_\mu{}^\alpha\theta^\mu + F^\alpha\theta \tag{4.8}$$

for some coefficients $C_\mu{}^\alpha$ and F^α, where

$$C_{\alpha\bar\beta} = \frac{i\omega_\mu{}^a{}_\alpha\omega^\mu{}_{a\bar\beta}}{n+2} - \frac{ig_{\alpha\bar\beta}\omega_\mu{}^a{}_\nu\omega^\mu{}_a{}^\nu}{2(n+1)(n+2)}. \tag{4.9}$$

In (4.9), we have used the vanishing of the curvature $\hat S_{A\bar B\nu\bar\mu}$ of the target quadric. We observe that the coefficients in front of θ^α and $\theta^{\bar\beta}$ in the pulled back forms $\hat\phi^\gamma$ are uniquely determined by the coframe (θ,θ^α) and the scalar products $\omega_\alpha{}^a{}_\mu\omega_{\bar\beta a\bar\nu}$. Hence, the second term on the right in (4.7), substituted in (4.6), contributes only terms of the form that appear on the right-hand side of (4.5). This completes the proof of Lemma 4.2. □

The final ingredient we shall need for the proof of our main result is the Gauss equation for the second fundamental form of the embedding. For our purposes, we shall only need the following form of it. A more general and precise version is stated and proved in [8] (Theorem 2.3; the lemma below corresponds to Eq. (7.17) in [8]). The proof in the Levi-nondegenerate case is identical to that of the strictly pseudoconvex case in [8], and is therefore not repeated here.

Lemma 4.3. *Let M, f, and $\omega_\alpha{}^a{}_\beta$ be as in Lemma 4.1. Then,*

$$0 = S_{\alpha\bar\beta\mu\bar\nu} + g_{a\bar b}\omega_\alpha{}^a{}_\mu\omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu} + T_{\alpha\bar\beta\mu\bar\nu}, \tag{4.10}$$

where $S_{\alpha\bar\beta\mu\bar\nu}$ is the Chern–Moser pseudoconformal curvature of M and $T_{\alpha\bar\beta\mu\bar\nu}$ is a conformally flat tensor.

5. Proof of Theorem 1.1

The first step in the proof of Theorem 1.1 is the following result concerning the second fundamental form and its derivatives. The notation is the same as in the previous sections. (For convenience of notation in the proof, we use f and $\tilde f$ to denote the mappings, rather than f_0 and f as in Theorem 1.1.) To simplify the notation, in what follows, we will use the notation ω_α^a, for $a \in \{1,\ldots,N-n\}$, instead of ω_α^{a+n} (and similarly for $\tilde\omega_\alpha^a$).

Theorem 5.1. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth Levi-nondegenerate hypersurface of signature $l \leqslant n/2$ and $p \in M$. Let $f : M \to Q_l^N$ and $\tilde f : M \to Q_l^{\tilde N}$ be smooth CR mappings that are CR transversal to Q_l^N at $f(p)$ and $Q_l^{\tilde N}$ at $\tilde f(p)$, respectively. Suppose that $N - n < l$ and $\tilde N \geqslant N$. Fix an admissible coframe (θ,θ^α) on M and choose corresponding coframes (as given by Proposition 3.1) $(\theta,\theta^A)_{A=1,\ldots,N}$ and $(\tilde\theta,\tilde\theta^A)_{A=1,\ldots,\tilde N}$ on Q_l^N and $Q_l^{\tilde N}$ adapted to $f(M)$ and $\tilde f(M)$, respectively. Denote by $(\omega_\alpha{}^a{}_\beta)_{a=1,\ldots,N-n}$ and $(\tilde\omega_\alpha{}^a{}_\beta)_{a=1,\ldots,\tilde N-n}$ the second fundamental forms of f and $\tilde f$, respectively, relative to these coframes. Let $k \geqslant 2$ be an integer and assume that the spaces $E_j(q)$*

and $\tilde{E}_j(q)$, for $2 \leqslant j \leqslant k$, are of constant dimension for q near p. Then, possibly after a unitary change of $(\tilde{\theta}^a)$ near p, the following holds for $2 \leqslant j \leqslant k$:

$$\begin{cases} \tilde{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j} = \omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j}, & a = 1,\ldots,N-n, \\ \tilde{\omega}_{\gamma_1}{}^{i}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j} = 0, & i = N-n+1,\ldots,\tilde{N}-n. \end{cases} \tag{5.1}$$

Remark 5.2. If f and $\tilde{f}$ in Theorem 5.1 are assumed to be CR transversal to Q_l^N and $Q_l^{\tilde{N}}$ at $f(p)$ and $\tilde{f}(p)$, respectively, for every $p \in M$, then for any $k \geqslant 2$ the set of points $p \in M$ such that the spaces $E_j(q)$ and $\tilde{E}_j(q)$, for $2 \leqslant j \leqslant k$, are of constant dimension for q near p is open and dense in M.

Proof. Recall the normalization of the Levi forms given by Proposition 3.1. We think of $(\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j})_{a=1,\ldots,N-n}$ and $(\tilde{\omega}_{\gamma_1}{}^{b}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j})_{b=1,\ldots,\tilde{N}-n}$ as vectors in $\mathbb{C}^{N-n}$ and $\mathbb{C}^{\tilde{N}-n}$, respectively. Let e_j denote the dimension of $E_j(q)$, for q near p and $j = 2,\ldots,k$. We first make an initial unitary change of the θ^a, $a = 1,\ldots,N-n$, near p such that, for each $j = 2,\ldots,k$, we have

$$\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j} = 0, \quad a = e_j+1,\ldots,N-n. \tag{5.2}$$

We then embed $\mathbb{C}^{N-n}$ in $\mathbb{C}^{\tilde{N}-n}$ as the subspace $\{W \in \mathbb{C}^{\tilde{N}-n}\colon\ W_i = 0,\ i = N-n+1,\ldots,\tilde{N}-n\}$, i.e. we extend $\omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j}$ to be 0 for $a = N-n+1,\ldots,\tilde{N}-n$. The proof now consists of showing that, possibly after a unitary change of the $\tilde{\theta}^a$, we have

$$\tilde{\omega}_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j} = \omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3,\ldots,\gamma_j}, \quad a = 1,\ldots,\tilde{N}-n. \tag{5.3}$$

If we subtract the Gauss equations for $\omega_\alpha{}^a{}_\beta$ given by (4.10) from the corresponding one for $\tilde{\omega}_\alpha{}^a{}_\beta$, we obtain (since the pseudoconformal curvature $S_{\alpha\bar{\beta}\mu\bar{\nu}}$ in both equations is computed using the same coframe (θ,θ^α))

$$-\sum_{a=1}^{N-n} \omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} + \sum_{b=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^b{}_\mu \tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}} = T'_{\alpha\bar{\beta}\mu\bar{\nu}}, \tag{5.4}$$

where $T'_{\alpha\bar{\beta}\mu\bar{\nu}}$ is a conformally flat tensor. For brevity, we will write this simply as

$$-\sum_{a=1}^{N-n} \omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} + \sum_{b=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^b{}_\mu \tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}} = 0 \quad \text{mod CFT}. \tag{5.5}$$

Let $\zeta := (\zeta^1,\ldots,\zeta^n)$, multiply (5.4) by $\zeta^\alpha\overline{\zeta^\beta}\zeta^\mu\overline{\zeta^\nu}$ and sum. Since the right-hand side of (5.4) is conformally flat, we obtain (see the beginning of Section 4)

$$-\sum_{a=1}^{N-n} \left|\omega^a(\zeta)\right|^2 + \sum_{b=1}^{\tilde{N}-n} \left|\tilde{\omega}^b(\zeta)\right|^2 = A(\zeta,\bar{\zeta})\left(-\sum_{i=1}^{l} |\zeta^i|^2 + \sum_{j=l+1}^{n} |\zeta^j|^2\right), \tag{5.6}$$

where $\omega^a(\zeta) = \omega_\alpha{}^a{}_\beta \zeta^\alpha \zeta^\beta$, $\tilde{\omega}^b(\zeta) = \tilde{\omega}_\alpha{}^b{}_\beta \zeta^\alpha \zeta^\beta$, and $A(\zeta, \bar{\zeta})$ is a polynomial in ζ and $\bar{\zeta}$. Recall that $N - n < l$. By Lemma 2.2, we conclude that $A \equiv 0$ and, hence,

$$\sum_{a=1}^{N-n} \left|\omega^a(\zeta)\right|^2 = \sum_{b=1}^{\tilde{N}-n} \left|\tilde{\omega}^b(\zeta)\right|^2, \tag{5.7}$$

or equivalently, since $\omega_\alpha{}^a{}_\mu = 0$ for $a = N - n + 1, \ldots, \tilde{N} - n$,

$$\sum_{a=1}^{\tilde{N}-n} \omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} = \sum_{b=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^b{}_\mu \tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}, \tag{5.8}$$

i.e. the collection of vectors $(\omega_\alpha{}^a{}_\beta)_{a=1,\ldots,\tilde{N}-n}$ and $(\tilde{\omega}_\alpha{}^a{}_\beta)_{a=1,\ldots,\tilde{N}-n}$ have the same scalar products with respect to the standard scalar product in $\mathbb{C}^{\tilde{N}-n}$. Hence, after a unitary change of $\tilde{\theta}^a$ (smooth by the constant dimension assumption on $E_2(q)$), we may assume that

$$\omega_\alpha{}^a{}_\beta = \tilde{\omega}_\alpha{}^a{}_\beta \tag{5.9}$$

near p.

Next, we take a covariant derivative in the direction θ^{γ_1} in the Gauss equations for $\omega_\alpha{}^a{}_\beta$ and $\tilde{\omega}_\alpha{}^a{}_\beta$ respectively, and then subtract the two resulting equations. Since the covariant derivative of a conformally flat tensor stays conformally flat and the covariant derivative of the curvature tensor $S_{\alpha\bar{\beta}\mu\bar{\nu};\gamma}$ is the same in both equations, we obtain

$$-\sum_{a=1}^{N-n} \left(\omega_\alpha{}^a{}_{\mu;\gamma_1} \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} + \omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu};\gamma_1}\right) + \sum_{b=1}^{\tilde{N}-n} \left(\tilde{\omega}_\alpha{}^b{}_{\mu;\gamma_1} \tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}} + \tilde{\omega}_\alpha{}^b{}_\mu \tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu};\gamma_1}\right) = 0 \mod \mathrm{CFT}. \tag{5.10}$$

By Lemma 4.1, the covariant derivatives $\omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu};\gamma_1}$ and $\tilde{\omega}_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu};\gamma_1}$ are conformally flat (since $\omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu};\gamma_1} = \overline{\omega_\beta{}^a{}_{\nu;\bar{\gamma}_1}}$). Hence, by using (5.9), we obtain

$$\sum_{a=1}^{N-n} \left(\omega_\alpha{}^a{}_{\mu;\gamma_1} - \tilde{\omega}_\alpha{}^b{}_{\mu;\gamma_1}\right) \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} = 0 \mod \mathrm{CFT}. \tag{5.11}$$

Since $N - n < l \leqslant n/2$, we conclude, by using Lemma 2.1 in the same way we used Lemma 2.2 above, that in fact

$$\sum_{a=1}^{N-n} \left(\omega_\alpha{}^a{}_{\mu;\gamma_1} - \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1}\right) \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} = 0, \tag{5.12}$$

which in turn implies

$$\tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1} = \omega_\alpha{}^a{}_{\mu;\gamma_1}, \quad a = 1, \ldots, e_2. \tag{5.13}$$

We now take two covariant derivatives in the directions θ^{γ_1} and $\theta^{\bar\gamma_1}$ in the two Gauss equations and subtract the resulting equations. By again using the facts that covariant derivatives of the form $\omega_a{}^a{}_{\beta;\bar\gamma}$ are conformally flat and covariant derivatives of conformally flat tensors are conformally flat, we obtain

$$\begin{aligned} &-\sum_{a=1}^{N-n}\big(\omega_\alpha{}^a{}_{\mu;\gamma_1\bar\gamma_1}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu}+\omega_\alpha{}^a{}_{\mu;\gamma_1}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\gamma_1}\big)\\ &\quad+\sum_{b=1}^{\tilde N-n}\big(\tilde\omega_\alpha{}^b{}_{\mu;\gamma_1\bar\gamma_1}\tilde\omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu}+\tilde\omega_\alpha{}^b{}_{\mu;\gamma_1}\tilde\omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu;\bar\gamma_1}\big)=0 \quad \operatorname{mod} \mathrm{CFT}. \end{aligned} \tag{5.14}$$

By Lemma 4.2, we have

$$\omega_\alpha{}^a{}_{\mu;\gamma_1\bar\gamma_1}=\omega_\alpha{}^a{}_{\mu;\bar\gamma_1\gamma_1}+C^a{}_{\alpha\mu\gamma_1\bar\gamma_1}{}^{\tau\nu}{}_b\omega_\tau{}^b{}_\nu \quad \operatorname{mod} \mathrm{CFT}, \tag{5.15}$$

where the $C^a{}_{\alpha\mu\gamma_1\bar\gamma_1}{}^{\tau\nu}{}_b$ only depend on the coframe (θ,θ^α) and the second fundamental form $\omega_\alpha{}^a{}_\beta$. Since $\omega_\alpha{}^a{}_{\mu;\bar\gamma_1\gamma_1}$ is conformally flat, we conclude that

$$\omega_\alpha{}^a{}_{\mu;\gamma_1\bar\gamma_1}=C^a{}_{\alpha\mu\gamma_1\bar\gamma_1}{}^{\tau\nu}{}_b\omega_\tau{}^b{}_\nu \quad \operatorname{mod} \mathrm{CFT}. \tag{5.16}$$

The same argument applied to $\tilde\omega_\alpha{}^a{}_{\mu;\gamma_1\bar\gamma_1}$, using the equality (5.9), shows that

$$\tilde\omega_\alpha{}^a{}_{\mu;\gamma_1\bar\gamma_1}=C^a{}_{\alpha\mu\gamma_1\bar\gamma_1}{}^{\tau\nu}{}_b\omega_\tau{}^b{}_\nu \quad \operatorname{mod} \mathrm{CFT} \tag{5.17}$$

with the same $C^a{}_{\alpha\mu\gamma_1\bar\gamma_1}{}^{\tau\nu}{}_b$. Substituting these identities back in (5.14), we obtain

$$-\sum_{a=1}^{N-n}\omega_\alpha{}^a{}_{\mu;\gamma_1}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\gamma_1}+\sum_{b=1}^{\tilde N-n}\tilde\omega_\alpha{}^b{}_{\mu;\gamma_1}\tilde\omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu;\bar\gamma_1}=0 \quad \operatorname{mod} \mathrm{CFT}. \tag{5.18}$$

By using Lemma 2.2 as above, we find that in fact

$$-\sum_{a=1}^{N-n}\omega_\alpha{}^a{}_{\mu;\gamma_1}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\gamma_1}+\sum_{b=1}^{\tilde N-n}\tilde\omega_\alpha{}^b{}_{\mu;\gamma_1}\tilde\omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu;\bar\gamma_1}=0. \tag{5.19}$$

Since we already have (5.13), we conclude that there is a unitary change of the remaining $\tilde\theta^{e_2+1},\dots,\tilde\theta^{\tilde N-n}$ such that

$$\tilde\omega_\alpha{}^a{}_{\mu;\gamma_1}=\omega_\alpha{}^a{}_{\mu;\gamma_1}. \tag{5.20}$$

We notice that such a unitary change of the coframes does not affect (5.9).

We now complete the proof of Theorem 5.1 by induction, using the ideas above. We assume that

$$\tilde\omega_\alpha{}^a{}_{\beta;\gamma_1,\dots,\gamma_j}=\omega_\alpha{}^a{}_{\beta;\gamma_1,\dots,\gamma_j}, \quad a=1,\dots,\tilde N-n, \tag{5.21}$$

holds for all $0 \leqslant j \leqslant k$ with $k \geqslant 2$. We wish to prove that (5.21) holds for all $0 \leqslant j \leqslant k+1$, after possibly another unitary change of the $\tilde{\theta}^a$. We apply repeatedly covariant derivatives in the directions $\theta^{\gamma_1}, \ldots, \theta^{\gamma_{k+1}}$ to the Gauss equations for $\omega_\alpha{}^a{}_\beta$ and $\tilde{\omega}_\alpha{}^a{}_\beta$. We obtain, using as above the fact that $\omega_\alpha{}^a{}_{\beta;\bar{\gamma}}$ is conformally flat,

$$-S_{\alpha\bar{\beta}\mu\bar{\nu};\gamma_1,\ldots,\gamma_{k+1}} = \sum_{a=1}^{N-n} \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} \mod \mathrm{CFT}, \tag{5.22}$$

and

$$-S_{\alpha\bar{\beta}\mu\bar{\nu};\gamma_1,\ldots,\gamma_{k+1}} = \sum_{a=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} \tilde{\omega}_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} \mod \mathrm{CFT}. \tag{5.23}$$

Subtracting these two equations, using the fact that $\omega_\alpha{}^a{}_\beta = \tilde{\omega}_\alpha{}^a{}_\beta$ and Lemma 2.1 as above, we conclude that

$$\tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}} = \omega_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}}, \quad a = 1, \ldots, e_2. \tag{5.24}$$

We now differentiate the two equations (5.22) and (5.23) in the direction $\theta^{\bar{\lambda}_1}$. We obtain

$$\begin{aligned} &-S_{\alpha\bar{\beta}\mu\bar{\nu};\gamma_1,\ldots,\gamma_{k+1}\bar{\lambda}_1} \\ &\quad = \sum_{a=1}^{N-n} \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}\bar{\lambda}_1} \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} + \sum_{a=1}^{N-n} \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} \omega_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu};\bar{\lambda}_1} \mod \mathrm{CFT}, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} &-S_{\alpha\bar{\beta}\mu\bar{\nu};\gamma_1,\ldots,\gamma_{k+1}\bar{\lambda}_1} \\ &\quad = \sum_{a=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}\bar{\lambda}_1} \tilde{\omega}_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu}} + \sum_{a=1}^{\tilde{N}-n} \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} \tilde{\omega}_{\bar{\beta}}{}^{\bar{a}}{}_{\bar{\nu};\bar{\lambda}_1} \mod \mathrm{CFT}. \end{aligned} \tag{5.26}$$

We now use Lemma 4.2 repeatedly to commute the covariant derivative $\omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}\bar{\lambda}_1}$ in (5.25) to the conformally flat derivative $\omega_\alpha{}^a{}_{\mu;\bar{\lambda}_1\gamma_1,\ldots,\gamma_{k+1}}$. In doing so, we produce, according to Lemma 4.2, new conformally flat terms as well as covariant derivatives of the form

$$\left(C^a{}_{\alpha\mu\gamma_1\ldots\gamma_{j+1}\bar{\lambda}_1}{}^{\mu_1\ldots\mu_{j+2}}{}_b\, \omega_{\mu_1}{}^b{}_{\mu_2;\mu_3\ldots\mu_{j+2}}\right)_{;\gamma_{j+2}\ldots\gamma_{k+1}} \tag{5.27}$$

with $0 \leqslant j \leqslant k-1$ and

$$C^a{}_{\alpha\mu\gamma_1\ldots\gamma_{k+1}\bar{\lambda}_1}{}^{\mu_1\ldots\mu_{k+2}}{}_b\, \omega_{\mu_1}{}^b{}_{\mu_2;\mu_3\ldots\mu_{k+2}}. \tag{5.28}$$

We note that, since $C^a{}_{\alpha\mu\gamma_1\ldots\gamma_{j+1}\bar{\lambda}_1}{}^{\mu_1\ldots\mu_{j+2}}{}_b$ only depends on the second fundamental form, all terms of the form (5.27) and (5.28) depend only on covariant derivatives $\omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_j}$ up to order at most $j = k$. If we repeat this procedure with (5.26), then the new terms that appear are either conformally flat or, by the induction hypothesis, precisely the same terms (of the form (5.27) and (5.28)) that appear in (5.25). Hence, when we subtract the two equations (5.25) and (5.26) we obtain, using again the fact that $\tilde{\omega}_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1} = \omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1}$,

$$\sum_{a=1}^{N-n} \left(\omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} - \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}\right)\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1} = 0 \quad \mathrm{mod\ CFT}. \tag{5.29}$$

Now, by using Lemma 2.1 as above we conclude that

$$\tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}} = \omega_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}}, \quad a = 1,\ldots,e_3. \tag{5.30}$$

We now apply repeated derivations in the directions $\theta^{\bar\lambda_2},\ldots,\theta^{\bar\lambda_k}$ to the two equations (5.25) and (5.26) and repeat the procedure and arguments above. The conclusion is that

$$\tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}} = \omega_\alpha{}^a{}_{\mu;\gamma_1\ldots\gamma_{k+1}}, \quad a = 1,\ldots,e_k. \tag{5.31}$$

The details of this are left to the reader.

In the final step, we apply a derivation in the direction $\theta^{\bar\lambda_{k+1}}$. After repeating the procedure above and subtracting the resulting equations we obtain

$$-\sum_{a=1}^{N-n} \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1\ldots\bar\lambda_{k+1}} + \sum_{a=1}^{N-n} \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}\tilde{\omega}_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1\ldots\bar\lambda_{k+1}} = 0 \quad \mathrm{mod\ CFT}. \tag{5.32}$$

We apply Lemma 2.2 as above and conclude that in fact

$$\sum_{a=1}^{N-n} \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}\omega_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1\ldots\bar\lambda_{k+1}} = \sum_{a=1}^{N-n} \tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}\tilde{\omega}_{\bar\beta}{}^{\bar a}{}_{\bar\nu;\bar\lambda_1\ldots\bar\lambda_{k+1}} = 0. \tag{5.33}$$

It follows, by using also (5.31), that there is a unitary change of the $\tilde\theta^a$, with $a = e_k + 1,\ldots,\tilde N - n$, such that

$$\tilde{\omega}_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}} = \omega_\alpha{}^a{}_{\mu;\gamma_1,\ldots,\gamma_{k+1}}.$$

This completes the induction and, thus, the proof of Theorem 5.1. □

In view of the definition of constant (k,s)-degeneracy given at the end of Section 3 and Remark 5.2, we obtain the following as a corollary of Theorem 5.1:

Corollary 5.3. *Let M, p, f, $\tilde f$ be as in Theorem 5.1. Then, there is an open neighborhood U of p in M such that for q in an open dense subset of U, the mapping $\tilde f$ is constantly (k,s)-degenerate at q for some $k \geqslant 2$ and some s with $\tilde N - s \leqslant N$.*

To prove Theorem 1.1, we also need the following result (Theorem 5.4 below). The corresponding result in the strictly pseudoconvex case was stated and proved in [8] (Theorem 2.2

in that paper). The proof in the Levi nondegenerate case is identical, and is therefore not reproduced here. We embed $\mathbb{C}^{N+1}$ into $\mathbb{CP}^{N+1}$ in the standard way, i.e. as the open subset $\{[z_0 : z_1 : \ldots : z_{N+1}]\colon z_0 \neq 0\}$.

Theorem 5.4. *Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic connected Levi-nondegenerate hypersurface of signature $l \leqslant n/2$ and $f : M \to Q_l^N \cap \mathbb{C}^{N+1} \subset \mathbb{C}^{N+1}$ a CR mapping that is CR transversal to $Q_l^N \cap \mathbb{C}^{N+1}$ at $f(p)$ for $p \in M$. Assume that f is constantly (k,s)-degenerate near p for some k and s. If $N - n - s < n$, then there is an open neighborhood V of p in M such that $f(V)$ is contained in the intersection of $Q_l^N \cap \mathbb{C}^{N+1}$ with a complex plane $P \subset \mathbb{C}^{N+1}$ of codimension s.*

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We first observe that it suffices to show that $f = T \circ L \circ f_0$, where T and L are as in the statement of the theorem, in an open neighborhood of any point $p \in M$. Indeed, if $f = T \circ L \circ f_0$ holds on a nonempty open subset of U, then it holds on all of U since both sides are holomorphic mappings $U \to \mathbb{CP}^{N+1}$ and U is connected.

Let $\Pi \subset \mathbb{CP}^{N+1}$ be the hyperplane at infinity, i.e. given in homogeneous coordinates by $z_0 = 0$, and observe that Π is biholomorphically equivalent to $\mathbb{CP}^N$. We observe that $Q_l^N \cap \Pi$ is a real hypersurface isomorphic to the hyperquadric Q_{l-1}^{N-1} and, hence, has signature $l-1$. Since $f(U) \not\subset Q_l^N$, it follows from Theorem 5.7 in [2] that $f(U)$ cannot be contained in Π. For, if it were, then $f(M)$ would be contained in $Q_l^N \cap \Pi \cong Q_{l-1}^{N-1}$ and $f(U) \not\subset Q_l^N \cap \Pi$, contradicting the conclusion of Theorem 5.7 in [2]. We claim that there is a dense relatively open subset $M_0 \subset M$ such that $f(p) \subset Q_l^N \cap \mathbb{C}^{N+1}$ and f is CR transversal to Q_l^N at $f(p)$ for every $p \in M_0$. Indeed, the existence of M_0 follows from the remarks above and Theorem 1.1 in [2], since $M' = Q_l^N$ satisfies condition (1.2) of that theorem ($l \leqslant n/2 \leqslant n-1$ for $n \geqslant 2$; note that the conditions in Theorem 1.1 of the present paper are never satisfied when $n = 1$). A similar argument applies to the mapping f_0 and after restricting M_0 if necessary, we may assume that $f_0(p) \in Q_l^{N_0} \cap \mathbb{C}^{N_0+1}$, $f(p) \in Q_l^N \cap \mathbb{C}^{N+1}$ and that both maps are CR transversal to their target manifolds for every $p \in M_0$.

By Corollary 5.3 (with the roles of f and $\tilde{f}$ played by f_0 and f, respectively), we conclude that there is a nonempty open subset of M_0 on which f is constantly (k,s)-degenerate for some k and s with $N - s \leqslant N_0$. Since $N - n - s \leqslant N_0 - n < l \leqslant n/2 < n$, Theorem 5.4 implies that there exists a point $p_0 \in M_0$ and an open neighborhood V of p_0 in M_0 such that $f(V)$ is contained in the intersection of $Q_l^N \cap \mathbb{C}^{N+1}$ with a complex plane $P \subset \mathbb{C}^{N+1}$ of codimension s. Since $N - s \leqslant N_0$, P is of dimension $\leqslant N_0 + 1$. Without loss of generality (by enlarging P if necessary), we may assume that the dimension of P equals $N_0 + 1$. Since f is CR transversal to Q_l^N at $f(p_0)$, the plane P must be transversal to Q_l^N at $f(p_0)$. The intersection $Q_l^N \cap P$ is again a hyperquadric (inside P) and its signature cannot exceed l. Since $f : V \to Q_l^N \cap P$ is a CR mapping that is CR transversal to $Q_l^N \cap P$, we conclude that $Q_l^N \cap P$ is a hyperquadric whose signature cannot be less than l, and hence the signature of $Q_l^N \cap P$ equals l. Let $\tilde{P}$ be the projective plane in $\mathbb{CP}^{N+1}$ whose restriction to $\mathbb{C}^{N+1}$ is P. Also, let $\tilde{P}_0$ denote the projective plane of dimension $N_0 + 1$ given by

$$\tilde{P}_0 := \{[z_0 : z_1 : \ldots : z_{N+1}] \in \mathbb{CP}^{N+1}\colon z_{N_0+2} = \cdots = z_{N+1} = 0\}.$$

Since both intersections $\tilde{P}\cap Q_l^N$ and $\tilde{P}_0\cap Q_l^N$ are nondegenerate quadrics of signature l, there exists (by elementary linear algebra) an automorphism $S\in \mathrm{Aut}(Q_l^N)$ such that $S(\tilde{P})=\tilde{P}_0$. Hence, the holomorphic mappings $S\circ f$ and $L\circ f_0$, where L is the linear embedding given by (1.2), both send V (by further shrinking V if necessary) into the nondegenerate quadric of signature l in the (N_0+1)-dimensional subspace $\{z_{N_0+2}=\cdots=z_{N+1}=0\}\subset \mathbb{C}^{N+1}$, which we may identify with the hyperquadric $Q_l^{N_0}\cap\mathbb{C}^{N_0+1}$ in $\mathbb{C}^{N_0+1}$. Now, since $(N_0-n)+(N_0-n)<2l\leqslant n$ and M is not locally equivalent to the quadric $Q^n_{n/2}$, Theorem 1.6 in [9] implies that there is an automorphism $T'\in \mathrm{Aut}(Q_l^{N_0})$ such that $S\circ f=L\circ T'\circ f_0$. Hence, near p_0, we have $f=S^{-1}\circ L\circ T'\circ f_0$. The mapping $S^{-1}\circ L\circ T'$ is a holomorphic embedding $\mathbb{CP}^{N_0+1}\to\mathbb{CP}^{N+1}$ that sends $Q_l^{N_0}$ into Q_l^N. It follows from the hypotheses that the signature l of the quadric $Q_l^{N_0}$ cannot be $N_0/2$ and, hence, it follows from [3] that there is an automorphism $T\in\mathrm{Aut}(Q_l^N)$ such that $S^{-1}\circ L\circ T'=T\circ L$. Consequently, the identity $f=T\circ L\circ f_0$ holds in a neighborhood of p_0 in $\mathbb{C}^{n+1}$. This completes the proof of Theorem 1.1 in view of the remark at the beginning of the proof. □

Remark 5.5. The proof of Theorem 1.1 could also be completed without reference to [9] by suitably modifying the proof of Theorem 7.2 in [8] to the Levi nondegenerate situation.

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