

Poisson homogeneous spaces & Quantum symmetric pairs

Rutgers talk

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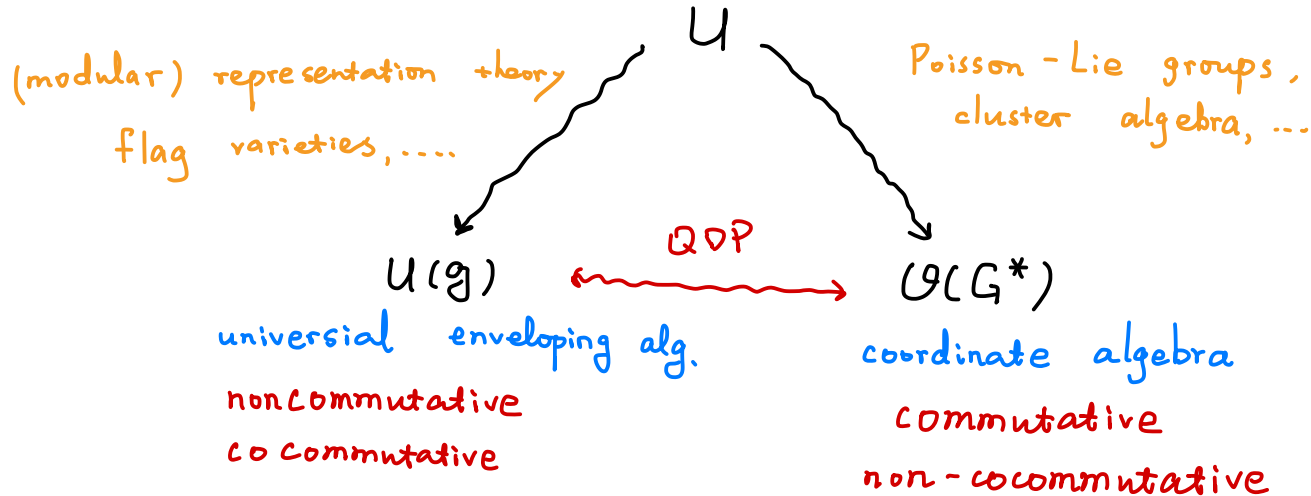
\mathfrak{g} : semi-simple Lie algebra / \mathbb{C}

$U = U_{\mathfrak{g}}(\mathfrak{g})$: Drinfeld - Jimbo quantum group: Hopf algebra / $\mathbb{C}(q)$

Quantum duality
principle (QDP)

(Drinfeld, De-Concini - Kaz - Procesi)

non commutative
non-cocommutative



$\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ Lie algebra involution

$$\mathfrak{k} = \mathfrak{g}^\theta,$$

(Letzter)

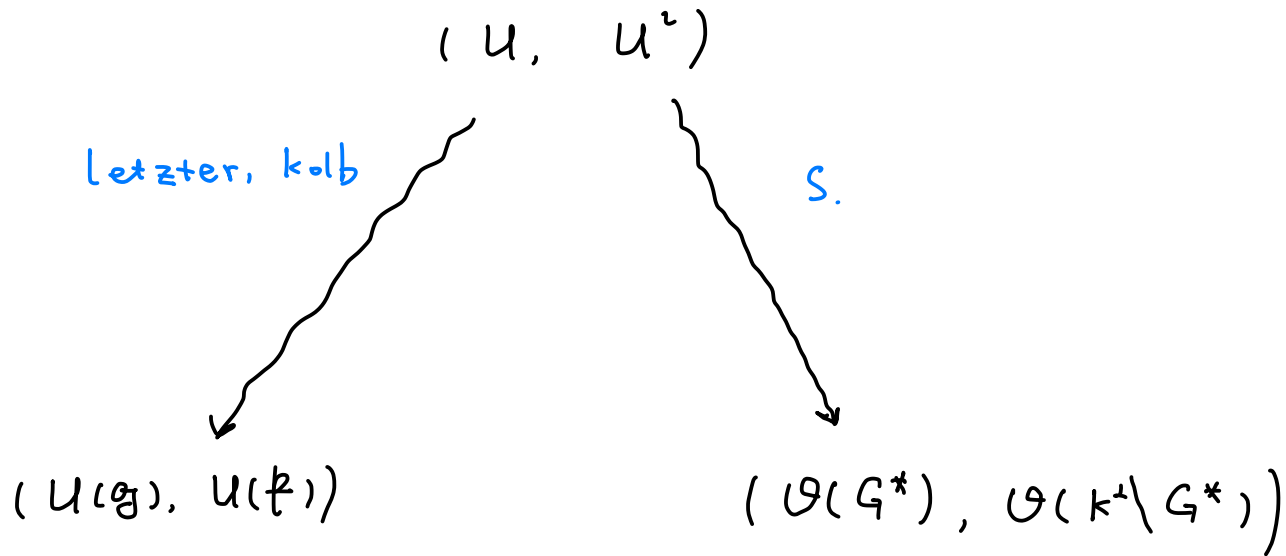
$U^\pm \subseteq U = U_\hbar(\mathfrak{g})$: quantum group : subalgebra of U

$$\Delta: U^\pm \rightarrow U^\pm \otimes U$$

(U, U^\pm) : quantum symmetric pair

E.g. $(U \otimes U, U)$: quantum symmetric pair of diagonal type

\rightsquigarrow quantum group is a generalisation of quantum group



k -orbits on flag variety (Bao-S. 22)

Symmetric space $k \backslash G$ (Bao-S. 24)

cluster realisation of U^v (S. 23)

§ Poisson algebraic groups, Lie bialgebras

Def Let R be a \mathbb{C} -algebra. A **Poisson bracket** $\{\cdot, \cdot\}: R \times R \rightarrow R$ is a Lie bracket, such that

$$\{x, yz\} = \{x, y\}z + y\{x, z\}, \quad \forall x, y, z \in R.$$

An affine variety V is called a **Poisson variety** if $\mathcal{O}(V)$ is equipped with a Poisson bracket.

A morphism between Poisson varieties $\varphi: V \rightarrow V'$ is a **Poisson morphism** if $\varphi^\#: \mathcal{O}(V') \rightarrow \mathcal{O}(V)$ is Poisson.

Def A linear algebraic group G with a Poisson structure is called a **Poisson algebraic group** if

$$m: G \times G \longrightarrow G \quad \text{is Poisson.}$$

\implies Linearization: Lie bialgebras

$$G: \text{Poisson algebraic group} \quad \mathfrak{g} = T_e G$$

$\implies \mathfrak{g}^*$ also has a Lie algebra structure:

$$\xi_1, \xi_2 \in \mathfrak{g}^* = T_e^* G, \quad \text{take } f_1, f_2 \in \mathcal{O}(G), \quad \xi_i = (df_i)_e,$$
$$[\xi_1, \xi_2]^* := (d\{f_1, f_2\})_e$$

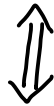
$$\Delta: \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g} \quad \text{cocommutator}$$

Def A Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \Delta)$ is a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ with $\Delta: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$, such that

$$\Delta[X, Y] = (\text{ad}_X \otimes 1 + 1 \otimes \text{ad}_X) \Delta(Y) - (\text{ad}_Y \otimes 1 + 1 \otimes \text{ad}_Y) \Delta(X)$$

\rightsquigarrow A duality:

$(\mathfrak{g}, [\cdot, \cdot], \Delta)$ is a Lie bialgebra



$(\mathfrak{g}^*, \Delta^*, [\cdot, \cdot]^*)$ is also a Lie bialgebra

• \mathfrak{g} : semi-simple Lie algebra

$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ Killing form

$(\mathfrak{b}_+, \mathfrak{b}_-)$: pair of Borel subalgebras, $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ is Cartan

• Nondegenerate invariant bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$:

$$((x_1, Y_1), (x_2, Y_2)) = \langle x_1, x_2 \rangle - \langle Y_1, Y_2 \rangle$$

Let $\mathfrak{g}_\Delta = \{ (x, x) \mid x \in \mathfrak{g} \} \cong \mathfrak{g}$

$$\begin{aligned} \mathfrak{p} &= \{ (x + H, -H + Y) \mid x \in \mathfrak{n}_+, Y \in \mathfrak{n}_-, H \in \mathfrak{h} \} \\ &\subseteq \mathfrak{b}_+ \oplus \mathfrak{b}_- \end{aligned}$$

claim $\mathfrak{g} = \mathfrak{g}_\Delta \oplus \mathfrak{p}$ as vector space with $(\mathfrak{g}_\Delta, \mathfrak{g}_\Delta) = 0$, $(\mathfrak{p}, \mathfrak{p}) = 0$.

$\Rightarrow \mathfrak{g}$ has a Lie bialgebra structure with $\mathfrak{g}^* \cong \mathfrak{p}$.

§ Quantization

• Quantized coordinate algebra

$$q : \text{indeterminate} \quad A = \mathbb{C}[q^{1/2}, q^{-1/2}]$$

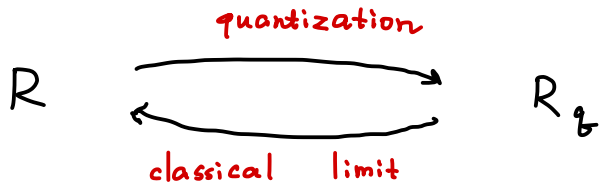
R_q : unital associative non-commutative A -alg

$$R := \mathbb{C} \otimes_{q \rightarrow 1} R_q \quad \text{Assume: } R \text{ is commutative}$$

$$\iff [R_q, R_q] \subseteq (q-1)R_q$$

$\implies \exists$ Poisson bracket on R :

$$\{ \bar{f}, \bar{g} \} = \frac{1}{q-1} (fg - gf), \quad \forall f, g \in R_q.$$



Drinfeld: Can also quantize the Lie bialgebra

• Quantized universal enveloping algebra

$(\mathfrak{g}, [\cdot, \cdot], \Delta)$: Lie bialgebra

$U(\mathfrak{g}) \rightsquigarrow U_{\hbar}(\mathfrak{g})$: Hopf algebra, Δ : coproduct

$$\Delta := \lim_{\hbar \rightarrow 0} \frac{\Delta - \Delta^{\text{op}}}{\hbar} : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$$

is the cocommutator

Advantage: generators & relations for $U_{\hbar}\mathfrak{g}$.

E.g. \mathfrak{g} : semi-simple Lie algebra / \mathbb{C} with Lie bialgebra structure

Quantized universal enveloping algebra

$$U = U_{\mathfrak{q}}(\mathfrak{g}) = \mathbb{C}(\mathfrak{q}) \langle E_i, F_i, k_i^{\pm 1} \mid 1 \leq i \leq r \rangle$$

$$\cdot k_i E_j = \mathfrak{q}_i^{a_{ij}} E_j k_i, \quad k_i F_j = \mathfrak{q}_i^{-a_{ij}} F_j k_i$$

$$\cdot [E_i, F_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{\mathfrak{q}_i - \mathfrak{q}_i^{-1}}$$

• (\mathfrak{q} -Serre relations)

$$\sum_{s=0}^{-a_{ij}} \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{\mathfrak{q}_i} E_i^s E_j E_i^{-a_{ij}-s} = 0$$

...

§ Quantum duality principal

$(\mathfrak{g}, [\cdot, \cdot], \delta)$: Lie bialgebra

QDP (Drinfeld)

{ quantized universal enveloping
algebra of $(\mathfrak{g}, [\cdot, \cdot], \delta)$ }



{ quantized coordinate algebra
of $(\mathfrak{g}^*, \delta^*, [\cdot, \cdot]^*)$ }

Rmk: Drinfeld stated QDP for formal quantization,
i.e. over $\mathbb{C}[[\hbar]]$, ($q = e^{\hbar}$)

A precise formulation of QDP for semi-simple groups

G : semi-simple group of adjoint type $\rightsquigarrow \mathfrak{g}$

$(B_+, B_-) \rightsquigarrow (\mathfrak{b}_+, \mathfrak{b}_-)$, $H = B_+ \cap B_-$ maximal torus

(U_+, U_-) : unipotent radicals

$$\mathfrak{g}^* \cong \mathfrak{p} = \{ (x + H, -H + Y) \mid x \in \mathfrak{n}_+, Y \in \mathfrak{n}_-, H \in \mathfrak{h} \}$$
$$\subseteq \mathfrak{b}_+ \oplus \mathfrak{b}_-$$

$$G^* = \{ (u_+ t, t^{-1} u_-) \mid u_{\pm} \in U_{\pm}, t \in H \}$$

dual Poisson group of G

Thm (De Concini - Kaz - Procesi)

There exists an $\mathcal{A} = \mathbb{C}[\mathfrak{q}, \mathfrak{q}^{-1}]$ -subalgebra ${}_{\mathcal{A}}\mathcal{U}$ of \mathcal{U} ,

and a Poisson algebra isomorphism:

$$\varphi: \mathbb{C}\mathcal{U} = \mathbb{C} \otimes_{\mathfrak{q}} {}_{\mathcal{A}}\mathcal{U} \xrightarrow{\sim} \mathcal{O}(G^*)$$

Rmk. ${}_{\mathcal{A}}\mathcal{U}$ is NOT the Lusztig integral form generated by divided powers.

§ Quantum symmetric pairs

G : adjoint semi-simple group / \mathbb{C}

$\theta: G \rightarrow G$: algebraic group involution

(B_+, B_-) pair of Borel, $H = B_+ \cap B_-$

$\theta B_+ \cap B_+$ has minimal dimension

$\rightsquigarrow (I = I_+ \cup I_0, \tau)$: Satake diagram

Def. (Letzter) The associated quantum symmetric pair (U, U^τ) consists of the quantum group U , and the quantum group U^τ , generated

by $F_i + c_i T_{w_i} (E_{\alpha_i}) k_i^{-1}$ ($i \in I_0$), $k_i = k_{\alpha_i}^{-1}$, ($i \in I_0$)

$F_i, E_i, k_i^{\pm 1}$ ($i \in I_+$)

Rmk. • U^+ is a vast generalisation of quantum group

• (Letzter, Kolb) A proper specialisation of (U, U^+) gives $(U(\mathfrak{g}), U(\mathfrak{k}))$.

• U^+ has generators and relations.

QDP for quantum symmetric pairs

Lemma Let $\mathfrak{k} = \mathfrak{g}^\theta$. Then

$$\mathfrak{k}^\perp = \{\xi \in \mathfrak{g}^* \mid \xi(\mathfrak{k}) = 0\} \subseteq \mathfrak{g}^* \cong \mathfrak{p}$$

is a Lie subalgebra.

Let $K^\perp \subseteq G^*$ be the connected closed subgroup with Lie algebra \mathfrak{k}^\perp .

Claim $K^\perp \backslash G^*$ is an affine Poisson homogeneous space of G^*

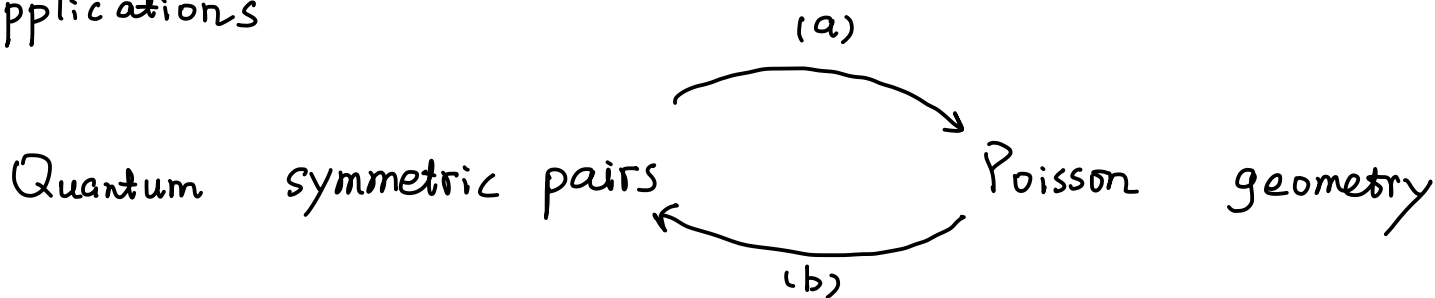
$\Rightarrow \mathcal{U}(K^\perp \backslash G^*) \hookrightarrow \mathcal{U}(G^*)$ is a Poisson subalgebra

Thm (S.) Let ${}_{\mathcal{A}}\mathcal{U}^* = {}_{\mathcal{A}}\mathcal{U} \cap \mathcal{U}^*$. There exists a Poisson algebra isomorphism $\varphi^*: {}_{\mathbb{C}}\mathcal{U}^* = \mathbb{C} \otimes {}_{\mathcal{A}}\mathcal{U}^* \xrightarrow{\sim} \mathcal{O}(k^+ \setminus G^*)$, such that the diagram

$$\begin{array}{ccc}
 \mathcal{O}(k^+ \setminus G^*) & \xrightarrow{\quad} & \mathcal{O}(G^*) \\
 \varphi^* \uparrow s & & s \uparrow \varphi \\
 {}_{\mathbb{C}}\mathcal{U}^* & \xrightarrow{\quad} & {}_{\mathbb{C}}\mathcal{U}
 \end{array}$$

commutes.

Applications



$$(a) \quad \mathfrak{g} = \mathfrak{sl}_n, \quad \theta: X \mapsto -X^T$$

$$G = \mathrm{PGL}_n, \quad K = \mathrm{PO}_n$$

$$K^+ \backslash G^* \cong (U_+, \{ \cdot, \cdot \}_{\mathrm{ou}})$$



Dubrovin-Ugaglia Poisson structure in 2d CFT

(b) By the construction of log-canonical coordinates on $\{\cdot, \cdot\}_{\text{ou}}$ [Chekhov-Shapiro], I constructed a cluster realisation $U^t \hookrightarrow \mathcal{U}_g(\mathcal{X}_{\Sigma_n})$.