# Poisson homogeneous spaces & Quantum symmetric pairs

Rutgers talk

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$\mathfrak{g}$ : semi-simple Lie algebra / $\mathbb{C}$

$U = U_q(\mathfrak{g})$ : Drinfeld-Jimbo quantum group : Hopf algebra / $\mathbb{C}(q)$

Quantum duality principle (QDP) (Drinfeld, De-Concini-Kac-Procesi)

noncommutative

non-cocommutative

(modular) representation theory

flag varieties, ....

Poisson-Lie groups,

cluster algebra, ...

$$\begin{tikzcd} & U \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ U(\mathfrak{g}) \arrow[rr, squiggly, leftrightarrow, "\text{QDP}"] & & \mathcal{O}(G^*) \end{tikzcd}$$

universal enveloping alg.

noncommutative

cocommutative

coordinate algebra

commutative

non-cocommutative

$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ Lie algebra involution

$\mathfrak{k} = \mathfrak{g}^{\theta}$,

(Letzter)

$\mathcal{U}^{\imath} \subseteq \mathcal{U} = \mathcal{U}_q(\mathfrak{g})$ : $\imath$quantum group : subalgebra of $\mathcal{U}$

$\Delta : \mathcal{U}^{\imath} \longrightarrow \mathcal{U}^{\imath} \otimes \mathcal{U}$

$(\mathcal{U}, \mathcal{U}^{\imath})$ : quantum symmetric pair

E.g. $(\mathcal{U} \otimes \mathcal{U}, \mathcal{U})$ : quantum symmetric pair of diagonal type

$\rightsquigarrow$ $\imath$quantum group is a generalisation of quantum group

$$
\begin{tikzcd}[column sep=small]
 & (\mathcal{U}, \mathcal{U}^{\imath}) \arrow[dl, squiggly, "\text{Letzter, Kolb}"'] \arrow[dr, squiggly, "\text{S.}"] & \\
(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{k})) & & (\mathcal{O}(G^*), \mathcal{O}(K^{\perp}\backslash G^*))
\end{tikzcd}
$$

$K$-orbits on flag variety (Bao-S. 22)

Symmetric space $K\backslash G$ (Bao-S. 24)

cluster realisation of $\mathcal{U}^{\imath}$ (S. 23)

## § Poisson algebraic groups, Lie bialgebras

**Def** Let $R$ be a $\mathbb{C}$-algebra. A Poisson bracket $\{\cdot,\cdot\} : R \times R \to R$ is a Lie bracket, such that

$$\{x, yz\} = \{x,y\}z + y\{x,z\}, \quad \forall\, x, y, z \in R.$$

An affine variety $V$ is called a Poisson variety if $\mathcal{O}(V)$ is equipped with a Poisson bracket.

A morphism between Poisson varieties $\varphi : V \to V'$ if

$$\varphi^{\#} : \mathcal{O}(V') \longrightarrow \mathcal{O}(V) \text{ is Poisson.}$$

**Def** A linear algebraic group $G$ with a Poisson structure is called a Poisson algebraic group if

$$m : G \times G \longrightarrow G \quad \text{is Poisson.}$$

$\rightsquigarrow$ Linearization : Lie bialgebras

$G$: Poisson algebraic group $\qquad \mathfrak{g} = T_e G$

$\rightsquigarrow$ $\mathfrak{g}^*$ also has a Lie algebra structure :

$\xi_1, \xi_2 \in \mathfrak{g}^* = T_e^* G$, take $f_1, f_2 \in \mathcal{O}(G)$, $\xi_i = (df_i)_e$,

$$[\xi_1, \xi_2]^* := (d\{f_1, f_2\})_e$$

$\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ cocommutator

**Def** A Lie bialgebra $(\mathfrak{g}, [\,,\,], \delta)$ is a Lie algebra $(\mathfrak{g}, [\,,\,])$ with $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, such that

$$\delta[X, Y] = (ad_X \otimes 1 + 1 \otimes ad_X)\,\delta(Y) - (ad_Y \otimes 1 + 1 \otimes ad_Y)\,\delta(X)$$

$\rightsquigarrow$ A duality :

$(\mathfrak{g}, [\,,\,], \delta)$ is a Lie bialgebra

$$\Updownarrow$$

$(\mathfrak{g}^*, \delta^*, [\,,\,]^*)$ is also a Lie bialgebra

- $\mathfrak{g}$ : semi-simple Lie algebra

  $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ killing form

$(\mathfrak{b}_+, \mathfrak{b}_-)$ : pair of Borel subalgebras, $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ is Cartan

- Nondegenerate invariant bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$:

$$\big((X_1, Y_1), (X_2, Y_2)\big) = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle$$

Let

$$\mathfrak{g}_\Delta = \{ (X, X) \mid X \in \mathfrak{g} \} \cong \mathfrak{g}$$

$$\mathfrak{p} = \{ (X + H, -H + Y) \mid X \in \mathfrak{n}_+, Y \in \mathfrak{n}_-, H \in \mathfrak{h} \}$$

$$\subseteq \mathfrak{b}_+ \oplus \mathfrak{b}_-$$

claim $\mathfrak{g} = \mathfrak{g}_\Delta \oplus \mathfrak{p}$ as vector space with $(\mathfrak{g}_\Delta, \mathfrak{g}_\Delta) = 0$, $(\mathfrak{p}, \mathfrak{p}) = 0$.

$\Longrightarrow$ $\mathfrak{g}$ has a Lie bialgebra structure with $\mathfrak{g}^* \cong \mathfrak{p}$.

# § Quantization

· Quantized coordinate algebra

$q$ : indeterminate $\qquad \mathcal{A} = \mathbb{C}[q^{1/2}, q^{-1/2}]$

$R_q$ : unital associative non-commutative $\mathcal{A}$-alg

$R := \mathbb{C} \otimes_{q \mapsto 1} R_q$ $\qquad$ Assume : $R$ is commutative

$\Longleftrightarrow [R_q, R_q] \subseteq (q-1) R_q$

$\rightsquigarrow$ $\exists$ Poisson bracket on $R$ :

$$\{\bar{f}, \bar{g}\} = \overline{\frac{1}{q-1}(fg - gf)}, \quad \forall f, g \in R_q .$$

$$R \underset{\text{classical limit}}{\overset{\text{quantization}}{\rightleftarrows}} R_q$$

Drinfeld: Can also quantize the Lie bialgebra

· Quantized universal enveloping algebra

$(\mathfrak{g}, [\,,\,], \delta)$ : Lie bialgebra

$U(\mathfrak{g}) \rightsquigarrow U_q(\mathfrak{g})$ : Hopf algebra, $\Delta$ : coproduct

$$\delta \text{ "="} \lim_{q \to 1} \frac{\Delta - \Delta^{op}}{q-1} : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$$

is the cocommutator

Advantage: generators & relations for $U_q\mathfrak{g}$.

E.g. $\mathfrak{g}$ : semi-simple Lie algebra / $\mathbb{C}$ with Lie bialgebra structure

Quantized universal enveloping algebra

$$\mathcal{U} = U_q(\mathfrak{g}) = \mathbb{C}(q)\langle E_i, F_i, K_i^{\pm 1} \mid 1 \le i \le r \rangle$$

- $K_i E_j = q_i^{a_{ij}} E_j K_i$ , $K_i F_j = q_i^{-a_{ij}} F_j K_i$

- $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

- ( $q$-Serre relations )

$$\sum_{s=0}^{-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} E_i^{s} E_j E_i^{-a_{ij}-s} = 0$$

...

## § Quantum duality principal

$(\mathfrak{g}, [\ ,\ ], \delta)$ : Lie bialgebra

QDP (Drinfeld)

$$\left\{ \begin{array}{c} \text{quantized universal enveloping} \\ \text{algebra of } (\mathfrak{g}, [\ ,\ ], \delta) \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{quantized coordinate algebra} \\ \text{of } (\mathfrak{g}^*, \delta^*, [\ ,\ ]^*) \end{array} \right\}$$

Rmk: Drinfeld stated QDP for formal quantization, i.e. over $\mathbb{C}[\![\hbar]\!]$, $(q = e^{\hbar})$

# A precise formulation of QDP for semi-simple groups

$G$ : semi-simple group of adjoint type $\rightsquigarrow \mathfrak{g}$

$(B_+, B_-) \rightsquigarrow (\mathfrak{b}_+, \mathfrak{b}_-)$ , $H = B_+ \cap B_-$ maximal torus

$(U_+, U_-)$ : unipotent radicals

$$\mathfrak{g}^* \cong \mathfrak{p} = \{ (X + H, -H + Y) \mid X \in \mathfrak{n}_+ , Y \in \mathfrak{n}_-, H \in \mathfrak{h} \}$$

$$\subseteq \mathfrak{b}_+ \oplus \mathfrak{b}_-$$

$$G^* = \{ (u_+ t, t^{-1} u_-) \mid u_\pm \in U_\pm , \ t \in H \}$$

dual Poisson group of $G$

<u>Thm</u> (De Concini - Kac - Procesi)

There exists an $\mathcal{A} = \mathbb{C}[q, q^{-1}]$-subalgebra ${}_{\mathcal{A}}\mathcal{U}$ of $\mathcal{U}$,

and a Poisson algebra isomorphism:

$$\varphi : \quad {}_{\mathbb{C}}\mathcal{U} = \mathbb{C} \otimes_{q \mapsto 1} {}_{\mathcal{A}}\mathcal{U} \xrightarrow{\sim} \mathcal{O}(G^*)$$

<u>Rmk</u>. ${}_{\mathcal{A}}\mathcal{U}$ is NOT the Lusztig integral form generated by divided powers.

## § Quantum symmetric pairs

$G$: adjoint semi-simple group / $\mathbb{C}$

$\theta : G \longrightarrow G$ : algebraic group involution

$(B_+, B_-)$ pair of Borel, $H = B_+ \cap B_-$

$\theta B_+ \cap B_+$ has minimal dimension

$\rightsquigarrow$ $(I = I_\bullet \sqcup I_\circ , \tau)$ : Satake diagram

**Def** (Letzter) The associated quantum symmetric pair $(\mathcal{U}, \mathcal{U}^\imath)$ consists of the quantum group $\mathcal{U}$, and the $\imath$quantum group $\mathcal{U}^\imath$, generated by

$$F_i + c_i T_{w_\bullet}(E_{\tau i}) K_i^{-1} \quad (i \in I_\circ), \quad K_i K_{\tau i}^{-1}, \quad (i \in I_\circ)$$

$$F_i, \quad E_i, \quad K_i^{\pm 1} \quad (i \in I_\bullet)$$

**Rmk.** • $U^\imath$ is a vast generalisation of quantum group

• (Letzter, Kolb) A proper specialisation of $(U, U^\imath)$ gives $(U(\mathfrak{g}), U(\mathfrak{k}))$.

• $U^\imath$ has generators and relations.

# QDP for quantum symmetric pairs

**Lemma** Let $\mathfrak{k} = \mathfrak{g}^\theta$. Then

$$\mathfrak{k}^\perp = \{ \xi \in \mathfrak{g}^* \mid \xi(\mathfrak{k}) = 0 \} \subseteq \mathfrak{g}^* \cong \mathfrak{p}$$

is a Lie subalgebra.

Let $K^\perp \subseteq G^*$ be the connected closed subgroup with Lie algebra $\mathfrak{k}^\perp$.

**Claim** $K^\perp \backslash G^*$ is an affine Poisson homogeneous space of $G^*$

$\Longrightarrow$ $\mathcal{O}(K^\perp \backslash G^*) \hookrightarrow \mathcal{O}(G^*)$ is a Poisson subalgebra

**Thm (S.)** Let ${}_{\mathcal{A}}U^\imath = {}_{\mathcal{A}}U \cap U^\imath$. There exists a Poisson algebra isomorphism $\psi^\imath : {}_{\mathbb{C}}U^\imath = \mathbb{C} \otimes {}_{\mathcal{A}}U^\imath \xrightarrow{\sim} \mathcal{O}(K^\perp \backslash G^*)$, such that the diagram

$$\begin{tikzcd}
\mathcal{O}(K^\perp \backslash G^*) \arrow[r, hook] & \mathcal{O}(G^*) \\
{}_{\mathbb{C}}U^\imath \arrow[u, "\psi^\imath", "\wr"'] \arrow[r] & {}_{\mathbb{C}}U \arrow[u, "\wr", "\psi"']
\end{tikzcd}$$

commutes.

# Applications

$$\text{Quantum symmetric pairs} \underset{(b)}{\overset{(a)}{\rightleftarrows}} \text{Poisson geometry}$$

(a) $\mathfrak{g} = \mathfrak{sl}_n$, $\quad \theta : X \longmapsto -X^T$

$G = PGL_n$ , $\quad K = PO_n$

$$K^{\perp} \backslash G^* \cong \left( U_+ , \{\cdot,\cdot\}_{DU} \right)$$

$\downarrow$

Dubrovin - Ugaglia Poisson structure in 2d CFT

(b) By the construction of log-canonical coordinates on $\{\cdot,\cdot\}_{DU}$ [Chekhov-Shapiro], I constructed a cluster realisation

$$U^{\iota} \hookrightarrow \mathcal{O}_q(\mathcal{X}_{\Sigma_n}).$$