Symmetric Functions. Talks

Def.

- Compositions: $\mathbb{N}^n$. $\mathbb{N}=\{0,1,\cdots\}$ $\quad S_n \curvearrowright \mathbb{N}^n$. w/ fund. domain

partition: $\lambda=(\lambda_1,\lambda_2,\cdots)$. $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$.

$|\lambda| = \sum \lambda_i$. $\quad \ell(\lambda) = \max_i \{\lambda_i \neq 0\}$.

$P_n = \{$ partitions of $n.\}$

- Symmetric polynomials & functions

$$\Lambda_n = \mathbb{Z}[x_1,\cdots,x_n]^{S_n}. \quad \Lambda_n = \bigoplus_{k\geq 0} \Lambda_n^k \text{ degree } k.$$

$m \geq n$. $\rho_{m,n} : \Lambda_m \to \Lambda_n$

$$f \mapsto f(x_1,\cdots,x_n,0,\cdots 0).$$

$\rho_{m,n}^k = \rho_{m,n}|_{\Lambda_m^k} : \Lambda_m^k \to \Lambda_n^k$. restriction

$\Lambda^k := \varprojlim_k \Lambda_n^k$. i.e. an elt in Λ^k is a seq. (f_n). s.t. $f_m(x_1,\cdots,x_n,0,\cdots,0) = f_n(x_1,\cdots,x_n)$.

$\Lambda := \bigoplus \Lambda^k$.

- Some bases:

$m_\lambda = \sum_{\alpha\sim\lambda} x^\alpha$. monomial symmetric functions

naturally a $\mathbb{Z}$-basis for Λ.

For $n\geq 1$. $e_0=1$. $e_{-n}=0$. $h_0=1$. $h_{-n}=0$. $P_0 =$ # variables.

$$e_n = m_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}.$$

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n} = \sum_{|\alpha|=n} x^\alpha = \sum_{|\lambda|=n} m_\lambda.$$

$$p_n = m_{(n)} = \sum_i x_i^n.$$

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_n}, \quad h_\lambda, \quad p_\lambda.$$

(free generators of Λ)

- Fact: Each of $\{e_n\}$, $\{h_n\}$, $\{p_n\}$ is algebraic independent.

  Each of $\{e_\lambda\}$, $\{h_\lambda\}$ is a $\mathbb{Z}$-basis for Λ.

  $\{p_\lambda\}$ is a $\mathbb{Q}$-basis for $\Lambda_\mathbb{Q} := \Lambda \otimes_\mathbb{Z} \mathbb{Q}$,

  e.g. $h_2 = \frac{1}{2}(p_1^2 + p_2)$.

- Generating functions:

$$E(t) = \sum_{n \ge 0} e_n t^n = \prod_{i \ge 1} (1 + x_i t)$$

$$H(t) = \sum_{n \ge 0} h_n t^n = \prod_{i \ge 1} (1 - x_i t)^{-1}$$

$$P(t) = \sum_{n \ge 0} p_{n+1} t^n = \sum_{n \ge 0} \sum_{i \ge 1} x_i^{n+1} t^n = \sum_{i \ge 1} \sum_{n \ge 0} x_i^{n+1} t^n$$

$$= \sum_{i \ge 1} \frac{x_i}{1 - x_i t} = \frac{d}{dt} \log H(t).$$

$$E(t) H(-t) = 1, \quad \overset{[t^n]}{\rightsquigarrow} \quad \sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0. \quad n \ge 1.$$

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}. \quad \overset{[t^n]}{\rightsquigarrow} \quad n h_n = \sum_{r=1}^{n} p_r h_{n-r}. \quad n \ge 1$$

$$P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}. \quad n e_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}.$$

$\Rightarrow$ $\{p_\lambda\}$ is a $\mathbb{Q}$-basis of $\Lambda_\mathbb{Q}$.

- The involution $\omega : \Lambda \to \Lambda$.

$$e_n \leftrightarrow h_n \qquad e_\lambda \leftrightarrow h_\lambda$$

$$p_n \leftrightarrow (-1)^{n-1} p_n \qquad p_\lambda \leftrightarrow \varepsilon_\lambda p_\lambda \qquad \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}$$

- $$H(t) = \exp \int P(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}.$$

$$E(t) = \sum z_\lambda^{-1} \varepsilon_\lambda p_\lambda t^{|\lambda|}.$$

$$z_\lambda := \prod_r (r^{m_r} \cdot m_r!), \quad \lambda = (1^{m_1} 2^{m_2} \cdots)$$

$$= |C_{S_n}(w)|$$, where the cycle type $\rho(w) = \lambda$.

$$H(t) = \exp\left(\sum_{n \geq 0} \frac{p_{n+1}}{n+1} t^{n+1} \right) = \prod_{r \geq 1} \exp\left(\frac{p_r}{r} t^r \right)$$

$$= \prod_{r \geq 1} \sum_{m_r \geq 0} \frac{p_r^{m_r}}{m_r! \, r^{m_r}} t^{r m_r}.$$

$$= \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}.$$

$$h_n = \sum_{|\lambda| = n} z_\lambda^{-1} p_\lambda \xrightarrow{\omega} e_n = \sum_{|\lambda| = n.} \varepsilon_\lambda z_\lambda^{-1} p_\lambda.$$

- Schur functions. (Fix $n \geq \ell(\lambda)$).

For $\alpha \in \mathbb{Z}^n$. define $a_\alpha := \det \left(x_i^{\alpha_j} \right)_{1 \leq i, j \leq n}$.

$$= \sum_{w \in S_n} \varepsilon(w) x^{w(\alpha)}.$$

a_α is skew-sym. i.e. $w a_\alpha = \varepsilon(w) \cdot a_\alpha$.

hence $a_\alpha = 0$ unless $\alpha_1, \cdots, \alpha_n$ are all distinct.

Assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$.

i.e. $\alpha = \lambda + \delta$. $\delta = (n-1, n-2, \cdots, 1, 0)$.

$a_{\lambda+\delta}$ is divisible by $(x_i - x_j)$. $(i<j)$. Hence by their product

$$\prod_{i<j} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta.$$

Define $s_\lambda := \dfrac{a_{\lambda+\delta}}{a_\delta}$. sym. homogeneous. degree $|\lambda|$.

$\{ s_\lambda \mid l(\lambda) \leq n \}$ forms a $\mathbb{Z}$-basis for Λ_n.

$\{ s_\lambda \mid |\lambda| = k \}$ forms a $\mathbb{Z}$-basis for Λ^k.

$\{ s_\lambda \}$ forms a $\mathbb{Z}$-basis for Λ.

- Jacobi-Trudi identity.

$$s_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j}).$$

It follows that $\omega : s_\lambda \leftrightarrow s_{\lambda'}$.

$$s_{(n)} = h_n. \quad s_{(1^n)} = e_n.$$

Proof. Let $e_r^{(k)} = e_r(x_1, \cdots, \hat{x}_k, \cdots x_n)$

Then $E^{(k)}(t) = \sum e_r^{(k)} t^r = \prod_{i \neq k} (1 + x_i t) = \dfrac{E(t)}{1 + x_k t}$.

$H(t) E^{(k)}(-t) = (1 - x_k t)^{-1}$

$\underset{\rightsquigarrow}{[t^{\alpha_i}]}$ $\displaystyle\sum_{j=1}^{n} h_{\alpha_i - n + j} (-1)^{n-j} e_{n-j}^{(k)} = x_k^{\alpha_i}$.

$\Rightarrow \quad H_\alpha M = A_\alpha$

where $H_\alpha = (h_{\alpha_i - n + j})$. $M = \left((-1)^{n-i} e_{n-i}^{(k)} \right)_{ik}$. $A = (x_j^{\alpha_i})$.

$$\det H_\alpha \det M = \det A_\alpha \qquad \alpha \in \mathbb{N}^n.$$

In particular, since $\det H_\delta = 1$. we have $\det M = \det A_\delta$.

$$\Rightarrow \qquad \det H_{\lambda+\delta} = \frac{\det A_{\lambda+\delta}}{\det A_\delta} = s_\lambda \qquad \square.$$

- Orthogonality. (Cauchy identity).

$$\prod_{i,j} (1-x_i y_j)^{-1} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y)$$
$$= \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y)$$
$$= \sum_\lambda s_\lambda(x) s_\lambda(y).$$

- View $x_i y_j$ as variable, then $\prod (1-x_i y_j)^{-1} = H(1) = \sum z_\lambda^{-1} p_\lambda(xy)$
$$= \sum z_\lambda^{-1} p_\lambda(x) p_\lambda(y).$$

- View x_i as variable, then $\prod (1-x_i y_j)^{-1} = \prod_j H(y_j)$
$$= \prod_j \sum_{\alpha_j=0}^{\infty} h_{\alpha_j}(x)\, y_j^{\alpha_j}$$
$$= \sum_{\alpha \in \mathbb{N}^n} h_\alpha(x)\, y^\alpha$$
$$= \sum_\lambda h_\lambda(x)\, m_\lambda(y).$$

- Assume $x = (x_1, \cdots, x_n)$. $y = (y_1, \cdots, y_n)$.

$$a_\delta(x)\, a_\delta(y) \prod (1-x_i y_j)^{-1}$$
$$= a_\delta(x) \sum_w \varepsilon(w)\, y^{w\delta} \sum_\alpha h_\alpha(x)\, y^\alpha$$
$$= a_\delta(x) \sum_{w,\alpha} \varepsilon(w)\, h_\alpha(x)\, y^{\alpha + w\delta}$$
$$\overset{\beta = \alpha + w\delta}{=} a_\delta(x) \sum_{w,\beta} \underbrace{\varepsilon(w)\, h_{\beta - w\delta}(x)}_{\sum_w \cdots = s_{\beta-\lambda}(x)}\, y^\beta.$$

$$= \sum_{\beta} a_\beta(x) y^\beta = \sum_{\lambda} \sum_{w} a_{\lambda+\delta}(x) \varepsilon(w) y^{w(\lambda+\delta)} \quad (\beta = w(\lambda+\delta))$$

$$= \sum_{\lambda} a_{\lambda+\delta}(x) a_{\lambda+\delta}(y) \qquad \square$$

Define a bilinear form on Λ by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$.

Prop. Let (u_λ), (v_λ) be $\mathbb{Q}$-bases of $\Lambda^n_{\mathbb{Q}}$ indexed by partitions of n.

TFAE:

(a). $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$.

(b). $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$. $\square$.

$\Rightarrow$ $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$. i.e. $\{p_\lambda\}$ is orthogonal. square norm z_λ.

$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ i.e. $\{s_\lambda\}$ is orthonormal.

$\langle \cdot, \cdot \rangle$ is sym. & pos. def.

ω is an isometry.

· The characters of the symmetric groups.

Each $w \in S_n$ can be uniquely written as a product of disjoint cycles, say of order $\rho_1 \geq \rho_2 \geq \cdots$. then $\rho(w) = (\rho_1, \rho_2, \cdots) \in P_n$. is called the cycle-type of w.

Embed $S_m \times S_n$ in S_{m+n}. all such embeddings are conjugate, hence the cycle-type of $v \times w$ is well-defined. and $\rho(v \times w)$
$= \rho(v) \cup \rho(w)$

Define $\psi|_{S_n}$: $S_n \to \Lambda^n$.
$w \mapsto p_{\rho(w)}$.

then $\psi(v \times w) = \psi(v)\, \psi(w)$.

$\leftarrow$ f.d.

= Grothendieck group of S_n-mod

Let $R^n = \mathbb{Z}$-span of $Irr(S_n)$ $\to$ {irr. characters of S_n}.
$(S_0 = \{1\}.\ R^0 = \mathbb{Z}\text{-span}\{1\}.)$

$$\operatorname{ind}_H^G \varphi = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1}).$$

$$R := \bigoplus_{n \geq 0} R^n$$

define a mult. on R by: $R^m \times R^n \to R^{m+n}$

$$f \cdot g = \operatorname{ind}_{S_m \times S_n}^{S_{m+n}} (f \times g).$$

$(f \times g)(v,w)$
$= f(v)\, g(w)$.

$\operatorname{ind}_H^G V$
$= \mathbb{C}G \otimes_{\mathbb{C}H} V$.

R is a comm. assoc. graded alg. with 1.

R has a scalar product: $\mathrm{Irr}\, S_n$ is an o.n.b. of R^n.

$$\langle f, g\rangle_{S_n} = \frac{1}{|S_n|}\sum_{w\in S_n} f(w) g(w^{-1}). \quad f, g\in R^n.$$

$$\langle f, g\rangle_R = \sum_{n\geq 0}\langle f_n, g_n\rangle_{S_n}. \quad f=\sum f_n\in R, \; g=\sum g_n\in R$$

Define the characteristic map.

$ch: R\to \Lambda_{\mathbb{C}} = \Lambda\otimes_{\mathbb{Z}}\mathbb{C}.$

$f\mapsto \langle f, \psi\rangle_R.$

$f\in R^n.$

$$ch(f) = \langle f, \psi\rangle_{S_n} = \frac{1}{n!}\sum_{w\in S_n} f(w)\psi(w) = \sum_{|\lambda|=n} z_\lambda^{-1} f_\lambda P_\lambda$$

$f_\lambda = f(w)$ where $\rho(w)=\lambda$.

$$\frac{|C(\lambda)|}{|S_n|} = \frac{1}{|C_{S_n}(\lambda)|} = z_\lambda^{-1}.$$

$u\in S_n.$ $u(i_1,\cdots,i_r)u^{-1} = (u(i_1),\cdots,u(i_r))$

Theorem. $ch: R\xrightarrow{\sim}\Lambda$ is an isometric isomorphism.

Proof. · Isometry $\langle ch(f), ch(g)\rangle_\Lambda$

$$= \left\langle \sum z_\rho^{-1} f_\rho P_\rho, \sum z_\rho^{-1} g_\rho P_\rho\right\rangle_\Lambda$$

$$= \sum z_\rho^{-1} f_\rho g_\rho = \langle f, g\rangle_{S_n}$$

· ring hom.

$$ch(f\cdot g) = \langle \mathrm{Ind}_{S_m\times S_n}^{S_{m+n}}(f\times g), \psi_{S_{m+n}}\rangle_{S_{m+n}}.$$

$$\underset{\text{Reciprocity}}{\overset{\text{Frobenius}}{=}} \langle f \times g, \psi|_{S_m \times S_n} \rangle_{S_m \times S_n}$$

$$= \langle f, \psi|_{S_m} \rangle_{S_m} \cdot \langle g, \psi|_{S_n} \rangle_{S_n}$$

$$= ch(f) \cdot ch(g).$$

- construct $\chi^\lambda \in R^n$. s.t. $ch(\chi^\lambda) = s_\lambda$.

Let η_n be the identity character of S_n (i.e. $\eta_n(w) = 1$).

then $\quad ch(\eta_n) = \sum_{|\rho|=n} z_\rho^{-1} p_\rho = h_n$

For $|\lambda| = n$. Let $\eta_\lambda = \eta_{\lambda_1} \cdot \eta_{\lambda_2} \cdots = \mathrm{Ind}_{S_\lambda}^{S_n}$ (identity character)

then $\quad ch(\eta_\lambda) = h_\lambda$.

Define $\chi^\lambda = \det(\eta_{\lambda_i - i + j}) \in R^n$

then $\quad ch(\chi^\lambda) = \det(ch(\eta_{\lambda_i - i + j}))$

$$= \det(h_{\lambda_i - i + j}) = s_\lambda.$$

- $\langle \chi^\lambda, \chi^\mu \rangle_{S_n} = \langle s_\lambda, s_\mu \rangle_\Lambda = \delta_{\lambda\mu}$.

hence χ^λ pairwise distinct, and up to sign, Irr. characters of S_n. But $|\mathrm{Irr}\, S_n| = |\text{Conj. classes of } S_n| = |P_n|$.

hence χ^λ exaust $\mathrm{Irr}\, S_n$, which is a $\mathbb{Z}$-basis of R^n

hence $ch : R^n \xrightarrow{\sim} \Lambda^n$. $\square$

Prop. $f^\lambda = \chi^\lambda(1) = K_{\lambda,(1^n)} = \#$ std. tab. of shape λ. > 0.

hence χ^λ is indeed an irr. character.

Proof. $s_\lambda = ch(\chi^\lambda) = \sum z_\rho^{-1} \chi_\rho^\lambda p_\rho$.

$$\xrightarrow{\langle \cdot, p_\rho \rangle_\Lambda} \quad \langle s_\lambda, p_\rho \rangle = \chi_\rho^\lambda.$$

in particular,

$$\chi^\lambda(1) = \chi^\lambda_{(1^n)} = \langle s_\lambda, p_{(1^n)} \rangle$$

$$= \langle s_\lambda, h_{(1^n)} \rangle = K_{\lambda,(1^n)}. \quad \square.$$

$s_\lambda = \sum K_{\lambda\mu} m_\mu$ so $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$.

Kostka number

Corollary: $$p_\rho = \sum_\lambda \chi_\rho^\lambda s_\lambda.$$

i.e. the transition mtx $M(p,s)$ is the character table. $\square$.

$$\left(\chi_\rho^\lambda = [x^{\lambda+\delta}]\, a_\delta p_\rho. \right)$$

Combinatorial formula / tab sum formula.

$$s_\lambda = \sum_T \prod_{s\in\lambda} x_{T(s)}.$$

$K_{\lambda\mu} = \#$ tab of shape λ wt μ.

$$\Rightarrow s_\lambda = \sum K_{\lambda\mu} m_\mu.$$

- $K_{\lambda,(1^n)} = \#\{\text{std tab of shape } \lambda\}$
$$= \frac{n!}{h(\lambda)}.$$

Proof 1. §1. Ex 1. §7. Ex 6.
Proof 2: §4. Ex 2,3. (⇐ §3. Ex 1 ⇐ §1. Ex 1,2,3, §2 Ex 3.)

- $\mathrm{ch}\,\chi^{(n)} = S_{(n)} = h_n = \mathrm{ch}(\eta_n)$, hence $\chi^{(n)} = \eta_n$.
identity character.

$e_n = \mathrm{ch}\,\chi^{(1^n)} = \langle \chi^{1^{(n)}}, \psi \rangle_{S_n} = \sum_{|\rho|=n} \chi_\rho^{1^{(n)}} z_\rho^{-1} p_\rho.$
$\overset{\shortparallel}{S_{(1^n)}} =$

$$e_n = \sum_{|\rho|=n} \varepsilon_\rho z_\rho^{-1} p_\rho. \quad \text{where } \varepsilon_\rho = |\rho| - \ell(\rho).$$
$$= \sum(\rho_i - 1).$$

hence $\varepsilon_n := \chi^{(1^n)}$ is the sign character.

(r-cycle has sign $(-1)^{r-1}$).

□

- $\chi_\rho^{\lambda'} = \varepsilon_n \chi^\lambda$ (pointwise product, afforded by tensor product of modules).

$\chi_\rho^{\lambda'} = \langle S_{\lambda'}, p_\rho \rangle$
$= \langle S_\lambda, \varepsilon_\rho p_\rho \rangle = \varepsilon_\rho \chi_\rho^\lambda.$