An Overview of Lie Superalgebras

Graduate Algebraic Representation Theory Seminar

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Structure

- Background
- Definitions & Examples
- Structure & Representation Theory
- Supersymmetry
- My recent work

Background

- In physics: Supersymmetry regarding particles of different statistics (Bosons & Fermions)

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- In mathematics: Graded Lie algebras in deformation theory

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Super Lie theory still has a lot of unsolved mysteries. It lives on as some pure mathematical pursuit.

General Principle of Superization

A (good) $\mathbb{Z}_2$-grading for everything!

$$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\} = \{\mathtt{even}, \mathtt{odd}\}$$

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Let's review Lie algebras!

Lie Algebras

Definition ([Hum78])

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew symmetric and satisfies the Jacobi identity, that is

1. $[X, Y] = -[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]$

Classic example: $\operatorname{End} V$ equipped with the usual commutator ($[A, B] = AB - BA$). As a Lie algebra, we denote it as $\mathfrak{gl}(V)$.

A bit of rep theory

Definition

Let $\mathfrak{g}$ be a Lie algebra. A representation is a pair (π, V) such that $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a linear map preserving the Lie bracket, i.e. $\pi([X, Y]) = [\pi(X), \pi(Y)]$. We say V is a $\mathfrak{g}$-module.

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"Redefine" $\mathfrak{g}$

Jacobian id. $\iff$ $\mathfrak{g}$ is a $\mathfrak{g}$-module.

Linear Superalgebra

Definition

A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A vector $v \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is said to be *even* (resp. *odd*) and write $|v| = 0$ (resp. 1). Denote the vector superspace over k with even subspace k^m and odd subspace k^n as $k^{m|n}$. Its *dimension* is denoted as $(m|n)$ while the *superdimension* is defined as $m - n$.

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A linear map $f : V \to W$ is *even* if it preserves parity, that is, $|f(v)| = |v|$ is even. If $|f(v)| = -|v|$, it is defined to be *odd*. An even or odd map is said to be *homogeneous*. We have

$$\begin{cases} \mathrm{Hom}(V, W)_{\bar{0}} := \mathrm{Hom}(V_{\bar{0}}, W_{\bar{0}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{1}}) \\ \mathrm{Hom}(V, W)_{\bar{1}} := \mathrm{Hom}(V_{\bar{0}}, W_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, W_{\bar{0}}) \end{cases}$$

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It means $\otimes$ is well-defined as follows:

$$(V \otimes W)_i := \bigoplus_{j+k=i} V_j \otimes W_k$$

with a natural isomorphism from $V \otimes W$ to $W \otimes V$:

$$s_{V,W} : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

Linear Superalgebra

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Linear Superalgebra

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$\left(\begin{array}{c|c} A_{m\times m} & 0_{m\times n} \\ \hline 0_{n\times m} & D_{n\times n} \end{array}\right)$ correspond to even linear maps, and

$\left(\begin{array}{c|c} 0_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & 0_{n\times n} \end{array}\right)$ to odd linear maps.

Lie Superalgebras

Definition ([Kac77])

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew supersymmetric and satisfies the super Jacobi identity:

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

Note:

1. Everything looks similar!
2. The sign can be explained by SVect.
3. $\mathfrak{g}_{\bar{0}}$ is just a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-mod.
4. The bracket is symmetric on $\mathfrak{g}_{\bar{1}}$.

Super $\mathfrak{gl}$

We write $\mathrm{End}(\mathbb{C}^{m|n})$ as $\mathfrak{gl}(m|n)$. As matrices: $\left(\begin{array}{c|c} A_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & D_{n\times n} \end{array}\right)$

$\mathfrak{gl}_{\bar{0}}$: $\left(\begin{array}{c|c} A_{m\times m} & 0_{m\times n} \\ \hline 0_{n\times m} & D_{n\times n} \end{array}\right)$

$\mathfrak{gl}_{\bar{1}}$: $\left(\begin{array}{c|c} 0_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & 0_{n\times n} \end{array}\right)$

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Classification

We aim to give an overview of the classification of complex, simple, and finite dimensional Lie superalgebras.
"Simple $\iff$ no non-trivial ideals" as usual.

Lie Algebras

7 types:

1. A_n, B_n, C_n, D_n: classical, (countably!) infinite families;
2. $E_{6,7,8}, F_4, G_2$: exceptional. Dimensions: 78, 133, 248, 52, 14

Lie Algebras

$\mathfrak{sl}(n+1)$ = the Lie algebra of $(n+1)\times(n+1)$ traceless matrices; *special linear Lie algebra*; $\boxed{A_n}$

$\mathfrak{so}(n) := \{X \in \mathrm{End}(k^n) : f(Xv, w) = -f(v, Xw), \forall v, w\}$, where f is a non-degenerate symmetric bilinear form; *orthogonal Lie algebra*; $\boxed{B_n}$ for $2n+1$ and $\boxed{D_n}$ for $2n$. VERY DIFFERENT DESPITE THEIR SIMILAR APPEARANCES

$\mathfrak{sp}(2n) := \left\{X \in \mathrm{End}(k^{2n}) : f(Xv, w) = -f(v, Xw), \forall v, w\right\}$, where f is a non-degenerate symplectic bilinear form. Non-degeneracy $\Rightarrow$ dim must be even; *symplectic Lie algebra*; $\boxed{C_n}$

Lie Algebras

The condition $f(Xv, w) = -f(v, Xw)$ comes from the corresponding Lie group condition

$$f(gv, gw) = f(v, w)$$

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with $g = e^{tX}$ and differentiating at $t = 0$.
Matrix form of f:

1. Orthogonal: $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$
2. Symplectic: $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

The condition: $fX = -X^\top f$

Classification Scheme

So what do with the LSAs?

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$$\text{Simple}\begin{cases}\underset{\mathfrak{g}_{\bar{1}}\text{ is a completely reducible}\,\mathfrak{g}_{\bar{0}}-\text{mod}}{\text{Classical}} & \begin{cases}\underset{\text{even non-deg. inv. form}}{\text{Basic}}{}^{1} \\ \text{Strange}\end{cases} \\ \underset{\text{weird ones}}{\text{The Cartan Series}} & \end{cases}$$

Good references: [Mus12, CW12]

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not so good reference:[Kac77]

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Special Linear LSA

Define the *supertrace* of $X = \left(\begin{array}{c|c} A_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & D_{n\times n} \end{array}\right)$, denoted $\mathrm{str}(X)$, as $\mathrm{tr}(A) - \mathrm{tr}(D)$.[2]

Define $\mathfrak{sl}(m|n) := \{X \in \mathfrak{gl}(m|n) : \mathrm{str}(X) = 0\}$. Guaranteed to be simple, except...

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when $m = n$, then $I_{n|n} = \mathrm{diag}(I_n, I_n) \in \mathfrak{sl}(n|n)$ which is central. We take the quotient to get $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\mathbb{C}I_{n|n}$.

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This is the Type A analog.

A

$A(m,n) := \mathfrak{sl}(m+1|n+1), m > n \geq 0$ and $A(n,n) := \mathfrak{psl}(n+1|n+1)$.

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Orthosymplectic LSA

Let's look at Type BCD analogs all at once.

Definition

Let $V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. A bilinear form $f : V \times V \to \mathbb{C}$ is said to be *even* if $f(V_i, V_{\bar{1}-i}) = 0$.

In terms of matrices, f has top-right and bottom-left blocks equal to 0 matrices.

Definition

A bilinear form f is said to be *supersymmetric* if $f(v \otimes w) = f(s_{V,V}(v \otimes w))$ [3] for any $v, w \in V$.

If f is even, then $f|_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric and $f|_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric.

[3] slight abuse of notation

Orthosymplectic LSA

Skew-symmetric + non-degeneracy = symplectic

$\mathfrak{osp}$

For $i \in \mathbb{Z}_2$, $\mathfrak{osp}(V)_i := \left\{X \in \mathrm{End}(V)_i : f(Xv, w) = -(-1)^{i|v|} f(v, Xw), \forall v, w\right\}$

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Matrix form of f:

1. $V = \mathbb{C}^{2m+1|2n}$: $\begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 0 & I_m & & \\ 0 & I_m & 0 & & \\ & & & 0 & I_n \\ & & & -I_n & 0 \end{pmatrix}$
2. $V = \mathbb{C}^{2m|2n}$: delete the first row and column.

The condition is now $fX = -X^{\mathsf{s}\top} f$ where $(-)^{\mathsf{s}\top} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A^\top & C^\top \\ -B^\top & D^\top \end{pmatrix}$ is called the *supertranspose*.

Orthosymplectic LSA

BCD

$B(m, n) := \mathfrak{osp}(2m+1|2n)$, $m \geq 0, n \geq 1$; $D(m, n) := \mathfrak{osp}(2m|2n)$, $m \geq 2, n \geq 1$; $C(n) := \mathfrak{osp}(2|2n-2)$, $n \geq 2$.

Exceptional LSAs (still basic!)

1. $D(2,1,\alpha)$, a (continuum) one-parameter family, dim $= (6|8)$;
2. $F(4)$, aka $AB(1|3)$, dim $= (24|16)$;
3. $G(3)$, aka $AG(1|2)$. dim $= (17|14)$.

$D(2,1,\alpha)$

Denote $\mathfrak{g} = D(2,1,\alpha)$. Let $\mathfrak{g}^{(i)} = \mathfrak{sl}(2)$ and $V^{(i)}$ be the standard representation of $\mathfrak{sl}(2)$ for $i = 1,2,3$. Then $\mathfrak{g}_{\bar{0}} := \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}$ and $\mathfrak{g}_{\bar{1}} := V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ (an irreducible $\mathfrak{g}_{\bar{0}}$-module).

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Not enough to define a Lie superalgebra. Need to define the *symmetric* bracket from $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}}$ to $\mathfrak{g}_{\bar{0}}$. Do that by using three parameters $\alpha_1, \alpha_2, \alpha_3$. Let $\mathfrak{g} = \mathfrak{g}(\alpha_1, \alpha_2, \alpha_3)$. Must have $\sum \alpha_i = 0$.

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Redundancy: $\mathfrak{g}(\alpha_1, \alpha_2, \alpha_3) = \mathfrak{g}(c\alpha_1, c\alpha_2, c\alpha_3) = \mathfrak{g}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)})$ for any non-zero $c \in \mathbb{C}$ and any $\sigma \in S_3$.

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It turns out that

$D(2,1,\alpha)$

$D(2,1,\alpha) := \mathfrak{g}(\alpha, 1, -1-\alpha)$

is simple when $\alpha \neq -1, 0$. Regarding the name, notice $D(2,1,1) = D(2,1) = \mathfrak{osp}(4|2)$.

$F(4)$ aka $AB(1|3)$, and $G(3)$ aka $AG(1|2)$

$F(4)_{\overline{0}} := \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$. $F(4)_{\overline{1}} :=$ Natural Rep. of $\mathfrak{sl}(2) \otimes$ Spin Rep. of $\mathfrak{so}(7)$.

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Details see [Mus12]

Strange Ones

The previous ones all have even non-deg. invariant bilinear forms (e.g. Killing forms). They are called *basic*.
The strange ones do NOT!

Periplectic LSA $\mathfrak{p}(n)$

Let $n \geq 2$. We let

$\mathfrak{p}(n)$

$$\mathfrak{p}(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) : \operatorname{tr} A = 0, B^\top = B, C^\top = -C \right\}.$$

$\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{sl}(n+1)$, and $\mathfrak{p}(n)_{\bar{1}} = \mathfrak{p}(n)_{-1} \oplus \mathfrak{p}(n)_1$ as a $\mathfrak{p}(n)_{\bar{0}}$-module. Here $\mathfrak{p}(n)_{\bar{0}}$ is the diagonal even part, while $\mathfrak{p}(n)_{\pm 1}$ are the B and C parts respectively.

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Can be regarded as the subalgebra of $\mathfrak{sl}(n+1|n+1)$ preserving certain odd symmetric form.

Queer LSA $\mathfrak{q}(n)$

Let $n \geq 2$. We let $\hat{\mathfrak{q}}(n) := \left\{ \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix} \right) \in \mathfrak{gl}(n+1|n+1) : \operatorname{tr} B = 0 \right\}$, and define

$\mathfrak{q}(n)$

$\mathfrak{q}(n) := [\hat{\mathfrak{q}}(n), \hat{\mathfrak{q}}(n)]/\mathbb{C}I_{n+1|n+1}$

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When people study $\mathfrak{q}(n)$, the $\hat{\mathfrak{q}}(n)$ version is often used for computations.

The Cartan Series

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$W(n), S(n), \tilde{S}(2n), H(n)$
Let $\bigwedge(n)$ be the exterior algebra on n letters ξ_i, $i = 1, \ldots, n$. $\bigwedge(n)$ has a natural parity grading induced by $\deg \xi_i = \overline{1}$. Then we define

$$W(n) := \operatorname{Der} \bigwedge(n)$$

with $W(n)_i = \{D \in \operatorname{End}_i(\bigwedge(n)) : D(ab) = D(a)b + (-1)^{i|a|} aD(b)\}$

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with $W(n)_i = \{D \in \operatorname{End}_i(\bigwedge(n)) : D(ab) = D(a)b + (-1)^{i|a|} aD(b)\}$
In particular, any homogeneous derivation can be expressed in the form of

$$\sum_{i=1}^{n} p_i \frac{\partial}{\partial \xi_i}$$

with $p_i \in \bigwedge(n)$. The other three are subalgebras of $W(n)$.

Classification Theorem

Theorem

The following is a complete list of finite dimensional simple Lie superalgebras over $\mathbb{C}$, up to some low rank isomorphisms:

1. *A finite dimensional simple Lie algebra;*
2. $A(m,n)$, $m > n \geq 0$; $A(n,n)$, $n \geq 1$; $B(m,n)$, $m \geq 0, n \geq 1$; $C(n)$, $n \geq 2$; $D(m,n)$, $m \geq 2, n \geq 1$ *(basic);*
3. $D(2,1,\alpha)$ *for* $\alpha \neq -1, 0$, $F(4)$, $G(3)$ *(exceptional, basic);*
4. $\mathfrak{p}(n)$, $\mathfrak{q}(n)$ *for* $n \geq 2$ *(strange);*
5. $W(n), S(n), \tilde{S}(2n), H(n)$ *(Cartan).*

Classification Theorem

Theorem

The following is a complete list of finite dimensional simple Lie superalgebras over $\mathbb{C}$, up to some low rank isomorphisms:

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Kac used non-degeneracy/degeneracy of Killing forms, rep. theory of $\mathfrak{g}_{\bar{0}}$, grading/filtration, etc. Pretty lengthy. Real forms and Kac–Moody superalgebras are studied by Vera Serganova [Ser83, Ser11].

Not so good news

1. Recall we may construct a semisimple Lie algebra using Cartan matrix + Serre's relations. The same can be said for the basic LSAs. But it fails for other LSAs.
2. A result by Djokovic and Hochschild says that the only not purely even LSAs with Weyl's complete reducibility is $\mathfrak{osp}(1|2n)$.
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Yes, I didn't talk about root systems but they exist.

Root Systems

We will look at Type A OF COURSE!

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The plot twist is the *form/inner product* on roots is different now. Remember we defined supertrace. Instead of $(\epsilon_i, \epsilon_j) = \delta_{ij}$, we now have

$$(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}, \ (\epsilon_i, \delta_j) = 0,$$

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where ϵ_i, δ_j are standard coordinates of the diagonal matrices.
Note $\epsilon_m - \delta_1$ has ODD root space and its length is 0. This means it's *isotropic*.

Supersymmetry

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Remember, $\epsilon_m - \delta_1$ is a simple root with length 0. The reflection is not well-defined.

Remark

For other simple roots, they generate $S_m \times S_n$ which just permutes ϵ's with ϵ's and δ's with δ's.

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3. The character formula gives a symmetric polynomial with respect to the Weyl group.
   Schur polynomials are symmetric in the usual sense as the Weyl group is S_n

Important question: how do we do these for $\mathfrak{gl}(m|n)$?

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   $\lambda' = \lambda - \alpha$ for $(\lambda, \alpha) \neq 0$ else $\lambda' = \lambda$. So $(\cdots, \overset{\times}{x}, \overset{\bullet}{y}, \cdots)$ becomes
   - $(\cdots, \overset{\bullet}{y}, \overset{\times}{x}, \cdots)$ if $x = -y$, or
   - $(\cdots, y \overset{\bullet}{+} 1, x \overset{\times}{-} 1, \cdots)$ if $x \neq -y$.
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3. The character formula gives a supersymmetric polynomial satisfying the usual symmetry + some additional properties.
   Yes, super Schur polynomials exist!

Supersymmetric Polynomials

Let V be an $m+n$ dimensional vector space with the standard basis $\epsilon_1,\dots,\epsilon_m,\delta_1,\dots,\delta_n$ and coordinates $x_1,\dots,x_m,y_1,\dots,y_n$. Let W_0 be $S_m\times S_n$ which acts on x_i and y_j separately. Let $f\in\mathfrak{P}(V)$ be a polynomial on V. In V, we set $\Pi_{\epsilon_i-\delta_j}:=\{v\in V : x_i(v)+y_j(v)=0\}$.
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We say f is *supersymmetric* if

1. $f \in \mathfrak{P}(V)^{W_0}$;
2. $f(X + \epsilon_i - \delta_j) = f(X)$ if $x_i + y_j = 0$, i.e. $f(X + \alpha) = f(X)$ for $X \in \Pi_{\alpha = \epsilon_i - \delta_j}$.

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The first condition is the usual symmetry, while the second one captures some "odd" condition.
Super Schur polynomials appear as characters of *certain* simple f.d. modules. They are supersymmetric and basis for the ring of supersymmetric polynomials.

Weyl Groupoids

Group = Small category of one object with invertible morphisms.
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How?

The object is sent to V, while a morphism is sent to a linear isomorphism of V.

Then how does a groupoid $\mathfrak{W}$ act on a vector space V?

Weyl Groupoids

Let $\mathcal{AF}(V)$ be the category in which

1. Objects: all affine subspaces of V;
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1. Objects: all affine subspaces of V;
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Let $\mathfrak{W}$ be a groupoid, then we say

Groupoid Action

$\mathfrak{W}$ acts on V if there is a functor $\mathtt{C}$ from $\mathfrak{W}$ to $\mathcal{AF}(V)$.

This degenerates to the usual group action if there is only one object $*$ and $\mathtt{C}(*) = V$.

Weyl Groupoids

Denote the set of isotropic roots as ${}_0\Sigma$. Let W_0 be the Weyl group which is generated by the reflections of anisotropic roots.

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The *isotropic roots groupoid* ${}_0\mathcal{S}$ is a groupoid such that $\mathrm{Obj}({}_0\mathcal{S}) = {}_0\Sigma$, with non-trivial morphisms $\bar{\tau}_\alpha : \alpha \to -\alpha$. Thus

$$\mathrm{Hom}_{{}_0\mathcal{S}}(\alpha, \beta) = \begin{cases} \varnothing & \text{if } \beta \neq \pm\alpha \\ \{\bar{\tau}_\alpha\} & \text{if } \beta = -\alpha \\ \{\mathrm{id}_\alpha\} & \text{if } \beta = \alpha \end{cases}.$$

One can define the semidirect product of W_0 and ${}_0\mathcal{S}$ via the action of W_0 on ${}_0\Sigma$. Let us define the Weyl groupoid as follows

$$\mathfrak{W} := W_0 \sqcup W_0 \ltimes {}_0\mathcal{S}$$

Weyl Groupoids

An action ([SV11]) of $\mathfrak{W}$ on $\mathfrak{h}^*$ is given by (loosely speaking),

1. sending $* \in \mathrm{Obj}(W_0)$ to the entire V, and W_0 acts on V as usual;
2. sending $\alpha \in \mathrm{Obj}({}_0\mathcal{S}) = {}_0\Sigma$ to $\Pi_\alpha := \{\mu \in \mathfrak{h}^* : (\mu, \alpha) = 0\}$, and $\bar{\tau}$ to $\tau : \mu \mapsto \mu + \alpha$ in Π_α;
3. making sure that W_0's action and ${}_0\mathcal{S}$'s action are compatible.

Weyl Groupoids

A function f on V is W-invariant if $f(wx) = f(x)$ for any $w \in W$. Similarly, we can define groupoid invariance:

Invariance

Let $\mathfrak{W}$ act on V via $\mathtt{C}$. Then a function F defined on V is said to be $\mathfrak{W}$-*invariant* if $F|_{\mathtt{C}(x)} = F|_{\mathtt{C}(y)} \circ \mathtt{C}(f)$ for any $f : x \to y$ in $\mathfrak{W}$.
Thus, $F(\mathtt{C}(f)x) = F(x)$.

Punchline

Supersymmetric polynomials on $\mathfrak{h}^*$ are $\mathfrak{W}$-invariant w.r.t. the action above.

My Work

I started with $\mathfrak{gl}$, but ended up with Type BC supersymmetry (even supersymmetry) as I used restricted root systems.

Set-up

Let's consider $\mathfrak{g} = \mathfrak{gl}(2m|2n), \mathfrak{k} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$. Such a pair comes from certain symmetric superspace.

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$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$$

where $\mathfrak{p}^\pm$ are abelian.
Turns out that as a $\mathfrak{k}$-module, $\mathfrak{U}(\mathfrak{p}^-) \otimes \mathfrak{U}(\mathfrak{p}^+)$ is completely reducible and multiplicity free. The components $W_\lambda^* \otimes W_\lambda$ are nicely parametrized by certain partitions/Young diagrams (λ).

Results

One may choose " $1 \in \mathrm{End}_{\mathfrak{k}}(W_\lambda)\mathrm{Id}_\lambda$' canonically.

$$\begin{array}{ccc} W_\lambda^* \otimes W_\lambda \hookrightarrow \mathfrak{U}\left(\mathfrak{p}^-\right) \otimes \mathfrak{U}\left(\mathfrak{p}^+\right) \to \mathfrak{U}\left(\mathfrak{g}\right)^{\mathfrak{k}} \xrightarrow{\Gamma} \mathfrak{S}(\mathfrak{a})^{\mathfrak{W}_0} \\ 1 \longmapsto D_\lambda \mapsto \Gamma(D_\lambda) \end{array}$$

Here Γ is the restricted root system version of Harish-Chandra isomorphism.

Results

Proposition ([Zhu22])

The algebra $\operatorname{Im}\Gamma$ *consists precisely of the symmetric polynomials on* $\mathfrak{a}^*$ *with Type BC supersymmetry property.*

Can be reformulated as

$$\operatorname{Im}\Gamma = \mathfrak{S}(\mathfrak{a})^{\mathfrak{W}} \cong \mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}}$$

Theorem ([Zhu22])

Assuming a conjecture, the Harish-Chandra image of the super Shimura operator associate with μ, $\Gamma(D_\mu)$, *is equal to some non-zero multiple of a Type BC supersymmetric interpolation polynomial.*

Thank you!

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