Differential and q-Difference Equations Characterizing Macdonald's Hypergeometric Series

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Joint work with Siddhartha Sahi

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper.

DISQUISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$

PARS PRIOR

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE SCIENTIARUM TRADITA 1812. JAN. 30.

Commentationes societatis regiae scientiarum Gottingensis recentiores Vol. II.
Gottingae MDCCCXIII.

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper.
In this paper, Gauss studied the *Gauss hypergeometric series*

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} \\ &\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}, \end{aligned}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$$

is the *Pochhammer symbol*. Assuming that $c \neq 0, -1, -2, \ldots$, Gauss proved that the series converges absolutely for $|z| < 1$, along with many other properties.

What interests us most is the following:

Theorem (Gauss, or Bateman Manuscript Project)

Gauss's ${}_2F_1(a, b; c; z)$ is the unique solution of the Euler's hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (c - (a+b+1)z)\frac{\mathrm{d}F}{\mathrm{d}z} - abF = 0, \quad F(0) = 1,$$

subject to the condition that $F(z)$ can be expressed as $\sum_{n=0}^{\infty} c_n z^n$.

${}_pF_q$

As a natural generalization, hypergeometric series with more parameters were studied thereafter:

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

Theorem (Bateman Manuscript Project)

The series ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; z)$ is the unique solution of

$$ \left(z\frac{\mathrm{d}}{\mathrm{d}z} \prod_{k=1}^{q} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1 \right) - z \prod_{k=1}^{p} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k \right) \right) (F) = 0, \quad F(0) = 1, $$

subject to the condition that $F(z)$ can be expressed as $\sum_{n=0}^{\infty} c_n z^n$.

The differential equation has two parts: diagonal ($z^n \to z^n$) and raising ($z^n \to z^{n+1}$). The DE is derived using combinatorics:

$$ \frac{(a)_{n+1}}{(a)_n} = a + n. $$

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Macdonald's series

Around 1988, Macdonald introduced the following series in his manuscripts, [arXiv:1309.4568, 1309.5208], associated with Jack and Macdonald polynomials in the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$,

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha) = \sum_\lambda \frac{(\boldsymbol{a}; \alpha)_\lambda}{(\boldsymbol{b}; \alpha)_\lambda} \alpha^{|\lambda|} \frac{J_\lambda(\boldsymbol{x}; \alpha)}{j_\lambda(\alpha)} $$

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; \alpha) = \sum_\lambda \frac{(\boldsymbol{a}; \alpha)_\lambda}{(\boldsymbol{b}; \alpha)_\lambda} \alpha^{|\lambda|} \frac{J_\lambda(\boldsymbol{x}; \alpha)}{j_\lambda(\alpha)} \frac{J_\lambda(\boldsymbol{y}; \alpha)}{J_\lambda(\boldsymbol{1}; \alpha)} $$

$$ {}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; q, t) = \sum_\lambda \frac{(\boldsymbol{a}; q, t)_\lambda}{(\boldsymbol{b}; q, t)_\lambda} t^{n(\lambda)} \frac{J_\lambda(\boldsymbol{x}; q, t)}{j_\lambda(q, t)} $$

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Here, the sums run over partitions with at most n parts, $(\cdot; \alpha)_\lambda$ and $(\cdot; q, t)_\lambda$ generalize the usual Pochhammer symbol $(\cdot)_n$, $J_\lambda(\alpha)$ and $J_\lambda(q, t)$ are Jack and Macdonald polynomials in the dual form and the unital form.

Differential and q-difference equations

- When $\boldsymbol{x}$ is a single variable z

$$\text{Jack series } {}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha) \longrightarrow \text{hypergeometric series } {}_pF_q(\boldsymbol{a};\boldsymbol{b};z)$$
$$\text{Macdonald series } {}_r\Phi_s(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};q,t) \longrightarrow q\text{-series } {}_r\phi_s(\boldsymbol{a};\boldsymbol{b};z;q)$$

Differential and q-difference equations for these series, for any p, q, r, s.

- When $\alpha = 2$, the zonal case, ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha = 2)$ has been studied in multivariate statistics since 1950s.
Differential equations for $p \leqslant 3$ and $q \leqslant 2$, by the work of [Muirhead 1970], [Constantine–Muirhead, 1972] and [Fujikoshi, 1975].

- In the general Jack and Macdonald case, for $p \leqslant 2$, $q \leqslant 1$: [Yan, 1992], [Kaneko, 1993], [Baker–Forrester 1997] and [Kaneko, 1996].

- In our recent papers, [C.–Sahi, 2510.10875] and [C., to be posted], we find differential and q-difference equations for the Jack and Macdonald case (resp.) for any p, q, r, s; unifying/generalizing all previous results.

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Our results

For the Jack case, we find differential operators $\mathcal{L}$ (lowering), $\mathcal{N}$ (diagonal), $\mathcal{R}$ (raising), depending on $\boldsymbol{a}, \boldsymbol{b}, \alpha$.

Theorem 1 ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha)$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\boldsymbol{0}) = 1. \tag{1}$$

Theorem 2 ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; \alpha)$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\boldsymbol{0}, \boldsymbol{0}) = 1. \tag{2}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_\lambda c_\lambda J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_\lambda c_\lambda J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha), \quad c_\lambda \in \mathbb{Q}(\alpha).$$

Also, we find a Macdonald analogue of the above:

differential operators $\longrightarrow$ q-difference operators

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The operators

The operators are derived using combinatorics:

$$\frac{(a;\alpha)_{\lambda\cup(i,j)}}{(a;\alpha)_\lambda} = a + \underbrace{j - 1 - \frac{i-1}{\alpha}}_{\text{the } \alpha\text{-content}}.$$

The diagonal operator $\mathcal{N}$ uses the *Debiard–Sekiguchi operator*

$$D(t) := \frac{1}{V(x)} \det\left(x_i^{n-j}\left(x_i\partial_i - (j-1)/\alpha + t\right)\right)_{1\leqslant i,j\leqslant n},$$

where $V(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. It acts diagonally on (J_λ) by

$$D(t)(J_\lambda) = \prod_i (\lambda_i - (i-1)/\alpha + t) \cdot J_\lambda.$$

The lowering and raising operators use the action of $e_1 = \sum x_i$ and $E_1 = \sum \frac{\partial}{\partial x_i}$, together with the *Laplace–Beltrami operator*.

Similarly, in the Macdonald case, we used the Macdonald operator, e_1, and (a q-difference analogue of) E_1.

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An Example

We have the binomial formula and the Cauchy identity for ${}_1F_0$:

$$F = {}_1F_0(a; \boldsymbol{x}; \alpha) = \sum_\lambda (a)_\lambda \frac{J_\lambda(\boldsymbol{x}; \alpha)}{j_\lambda(\alpha)} = \prod_{i=1}^n (1 - x_i)^{-a},$$

$$G = {}_1F_0(n/\alpha; \boldsymbol{x}, \boldsymbol{y}; \alpha) = \sum_\lambda \frac{J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha)}{j_\lambda(\alpha)} = \prod_{i,j=1}^n (1 - x_i y_j)^{-1/\alpha}.$$

In this case, the operators are

$$\mathcal{L}^{(\boldsymbol{x})} = \sum_i \partial_i, \quad \mathcal{N}^{(\boldsymbol{x})} = \sum_i x_i \partial_i, \quad \mathcal{R}^{(\boldsymbol{x})} = \sum_i x_i \left(x_i \partial_i + a \right),$$

and the Theorems read

$$\mathcal{N}^{(\boldsymbol{x})}(F) = F \cdot a \sum_i \frac{x_i}{1 - x_i} = \mathcal{R}^{(\boldsymbol{x})}(F),$$

$$\mathcal{L}^{(\boldsymbol{x})}(G) = G \cdot \frac{1}{\alpha} \sum_{i,j} \frac{y_j}{1 - x_i y_j} = \mathcal{R}^{(\boldsymbol{y})}(G).$$

Thank you!

I am currently on the postdoctoral job market.
Please feel free to contact me if you are interested!

- **Email:** hc813@math.rutgers.edu
- **Slides:** `https://sites.math.rutgers.edu/~hc813/`

- **Preprint:** arXiv:2510.10875