Hypergeometric series associated with Jack and Macdonald polynomials

Hong Chen

Rutgers

March 10, 2026
Ph.D. Dissertation Defense

Advisor: Prof. Siddhartha Sahi
arXiv: 2510.10875, 2602.13495

Hypergeometric series associated with Jack and Macdonald polynomials

Hong Chen

Rutgers

March 10, 2026
Ph.D. Dissertation Defense

Advisor: Prof. Siddhartha Sahi
arXiv: 2510.10875, 2602.13495

Function	Taylor Expansion	Differential Equation
$(1-z)^{-a}$	$\sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)\,k!}\, z^k$	$(1-z)F' - aF = 0$
$\frac{-\log(1-z)}{z}$	$\sum_{k=0}^{\infty} \frac{1}{k+1} z^k$	$z(1-z)F'' + (2-3z)F' - F = 0$
$\arcsin z$	$\sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)}\, z^{2k+1}$	$(1-z^2)F'' - zF' = 0$
$K(z)$	$\frac{\pi}{2} \sum_{k=0}^{\infty} \left(\frac{(2k)!}{2^{2k}(k!)^2} \right)^2 z^{2k}$	$z(1-z^2)F'' + (1-3z^2)F' - zF = 0$

Here $K(z)$ is the complete elliptic integral of the first kind:

$$K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z^2 \sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-z^2 t^2)}}.$$

Euler's differential equation

More generally, consider any second-order linear ODE:

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + a(z)\,\frac{\mathrm{d}F}{\mathrm{d}z} + b(z)F = 0, \quad a(z),\ b(z) \text{ meromorphic.}$$

By the work of Riemann (1857), if there are exactly three regular singularies in the equation, then it can be transformed into *Euler's differential equation*:

$$z(1-z)\,\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + (c-(a+b+1)z)\,\frac{\mathrm{d}F}{\mathrm{d}z} - abF = 0.$$

Euler (1769) gave a power series solution and presented an integral representation. (See Page 29.)

Euler's differential equation

More generally, consider any second-order linear ODE:

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + a(z)\,\frac{\mathrm{d}F}{\mathrm{d}z} + b(z)F = 0, \quad a(z),\ b(z) \text{ meromorphic.}$$

By the work of Riemann (1857), if there are exactly three regular singularities in the equation, then it can be transformed into *Euler's differential equation*:

$$z(1-z)\,\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + (c-(a+b+1)z)\,\frac{\mathrm{d}F}{\mathrm{d}z} - abF = 0.$$

Euler (1769) gave a power series solution and presented an integral representation. (See Page 29.)

Gauss' hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper.

DISQUISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$

PARS PRIOR

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE SCIENTIARUM TRADITA 1812. JAN. 30.

Commentationes societatis regiae scientiarum Gottingensis recentiores Vol. II.
Gottingae MDCCCXIII.

Gauss' hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper.
The *Gauss hypergeometric series/function* is

$$\begin{aligned} F(a,b;c;z) &= 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1) \cdot b(b+1)}{(1)(1+1) \cdot c(c+1)} z^2 \\ &\quad + \frac{a(a+1)(a+2) \cdot b(b+1)(b+2)}{(1)(1+1)(1+2) \cdot c(c+1)(c+2)} z^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1)_n (c)_n} z^n, \end{aligned} \tag{1}$$

where $(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)} = u(u+1)\cdots(u+n-1)$ is the *Pochhammer symbol*. In particular, $(1)_n = n!$.

Gauss' hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper.
The *Gauss hypergeometric series/function* is

$$\begin{aligned} F(a,b;c;z) &= 1 + \frac{a\cdot b}{1\cdot c}z + \frac{a(a+1)\cdot b(b+1)}{(1)(1+1)\cdot c(c+1)}z^2 \\ &\quad + \frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{(1)(1+1)(1+2)\cdot c(c+1)(c+2)}z^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1)_n (c)_n} z^n, \end{aligned} \tag{1}$$

where $(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)} = u(u+1)\cdots(u+n-1)$ is the *Pochhammer symbol*. In particular, $(1)_n = n!$.

$$(1-z)^{-a} = F(a,b;b;z), \qquad \arcsin z = z \cdot F(1/2,1/2;3/2;z^2),$$

$$\frac{-\log(1-z)}{z} = F(1,1;2;z), \qquad K(z) = \frac{\pi}{2} \cdot F(1/2,1/2;1;z^2).$$

Hypergeometric series, the classical

One natural way to generalize Gauss' $F(a, b; c; z)$ is to allow more parameters. Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$ (let $b_0 = 1$),

$$ {}_pf_q(\boldsymbol{a}; \boldsymbol{b}; z) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{a})_n}{(1, \boldsymbol{b})_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n. $$

This captures *all* series $\sum c_n z^n$ such that c_{n+1}/c_n is a rational polynomial in n:

$$ \frac{(n + a_1) \cdots (n + a_p)}{(n + b_0)(n + b_1) \cdots (n + b_q)}. $$

Question: Is there a differential equation that characterizes ${}_pf_q$?
Answer: Yes.

$$ \left(\prod_{k=0}^{q} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1 \right) - z \prod_{k=1}^{p} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + a_k \right) \right) (F) = 0, \quad F(0) = 1. $$

Hypergeometric series, the classical

One natural way to generalize Gauss' $F(a, b; c; z)$ is to allow more parameters. Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$ (let $b_0 = 1$),

$$ {}_pf_q(\boldsymbol{a}; \boldsymbol{b}; z) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{a})_n}{(1, \boldsymbol{b})_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n. $$

This captures *all* series $\sum c_n z^n$ such that c_{n+1}/c_n is a rational polynomial in n:

$$ \frac{(n + a_1) \cdots (n + a_p)}{(n + b_0)(n + b_1) \cdots (n + b_q)}. $$

Question: Is there a differential equation that characterizes ${}_pf_q$?

Answer: Yes.

$$ \left(\prod_{k=0}^{q} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1 \right) - z \prod_{k=1}^{p} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + a_k \right) \right) (F) = 0, \quad F(0) = 1. $$

Hypergeometric series, the classical

One natural way to generalize Gauss' $F(a, b; c; z)$ is to allow more parameters. Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$ (let $b_0 = 1$),

$$ {}_pf_q(\boldsymbol{a}; \boldsymbol{b}; z) \coloneqq \sum_{n=0}^{\infty} \frac{(\boldsymbol{a})_n}{(1, \boldsymbol{b})_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n. $$

This captures *all* series $\sum c_n z^n$ such that c_{n+1}/c_n is a rational polynomial in n:

$$ \frac{(n + a_1) \cdots (n + a_p)}{(n + b_0)(n + b_1) \cdots (n + b_q)}. $$

Question: Is there a differential equation that characterizes ${}_pf_q$?
Answer: Yes.

$$ \left(\prod_{k=0}^{q} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1 \right) - z \prod_{k=1}^{p} \left(z \frac{\mathrm{d}}{\mathrm{d}z} + a_k \right) \right) (F) = 0, \quad F(0) = 1. $$

Recall that

$$\frac{(u)_{n+1}}{(u)_n} = u + n, \tag{2}$$

then we have

$$\prod_{k=0}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1\right)(z^n) = \prod_{k=0}^{q}(n + b_k - 1)\cdot z^n = \frac{(1,\boldsymbol{b})_n}{(1,\boldsymbol{b})_{n-1}} z^n,$$

$$z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k\right)(z^n) = \prod_{k=1}^{p}(n + a_k)\cdot z^{n+1} = \frac{(\boldsymbol{a})_{n+1}}{(\boldsymbol{a})_n} z^{n+1}.$$

Then we have

$$z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k\right)({}_pf_q) = \sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{n+1}}{(\boldsymbol{a})_n}\cdot\frac{(\boldsymbol{a})_n}{(1,\boldsymbol{b})_n} z^{n+1} = \sum_{n=0}^{\infty}\frac{(1,\boldsymbol{b})_{n+1}}{(1,\boldsymbol{b})_n}\cdot\frac{(\boldsymbol{a})_{n+1}}{(1,\boldsymbol{b})_{n+1}} z^{n+1}$$

$$= 0 + \sum_{n=1}^{\infty}\frac{(1,\boldsymbol{b})_n}{(1,\boldsymbol{b})_{n-1}}\cdot\frac{(\boldsymbol{a})_n}{(1,\boldsymbol{b})_n} z^n = \prod_{k=0}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1\right)({}_pf_q).$$

Recall that

$$\frac{(u)_{n+1}}{(u)_n} = u + n, \tag{2}$$

then we have

$$\prod_{k=0}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1\right)(z^n) = \prod_{k=0}^{q}(n + b_k - 1)\cdot z^n = \frac{(1, \boldsymbol{b})_n}{(1, \boldsymbol{b})_{n-1}} z^n,$$

$$z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k\right)(z^n) = \prod_{k=1}^{p}(n + a_k)\cdot z^{n+1} = \frac{(\boldsymbol{a})_{n+1}}{(\boldsymbol{a})_n} z^{n+1}.$$

Then we have

$$\begin{aligned} z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k\right)({}_pf_q) &= \sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{n+1}}{(\boldsymbol{a})_n}\cdot\frac{(\boldsymbol{a})_n}{(1, \boldsymbol{b})_n} z^{n+1} = \sum_{n=0}^{\infty}\frac{(1, \boldsymbol{b})_{n+1}}{(1, \boldsymbol{b})_n}\cdot\frac{(\boldsymbol{a})_{n+1}}{(1, \boldsymbol{b})_{n+1}} z^{n+1} \\ &= 0 + \sum_{n=1}^{\infty}\frac{(1, \boldsymbol{b})_n}{(1, \boldsymbol{b})_{n-1}}\cdot\frac{(\boldsymbol{a})_n}{(1, \boldsymbol{b})_n} z^n = \prod_{k=0}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1\right)({}_pf_q). \end{aligned}$$

Basic hypergeometric series

In 1846, Heine introduced the following q-deformation:
The *q-Pochhammer symbol* $(u;q)_n$ is defined so that

$$\frac{(u;q)_{n+1}}{(u;q)_n} = 1 - uq^n. \tag{3}$$

Namely,

$$(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1}).$$

The *basic hypergeometric series* is

$${}_r\phi_s(\boldsymbol{a};\boldsymbol{b};z;q) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{a};q)_n}{(q,\boldsymbol{b};q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} z^n. \tag{4}$$

As $q \to 1$, since $\frac{(q^u;q)_n}{(1-q)^n} \to (u)_n$, we have

$${}_r\phi_s(q^{\boldsymbol{a}};q^{\boldsymbol{b}};(1-q)^{r-s-1}z;q) \to {}_rf_s(\boldsymbol{a};\boldsymbol{b};z).$$

There is a similar q-difference equation characterization of ${}_r\phi_s$.

Basic hypergeometric series

In 1846, Heine introduced the following q-deformation:
The *q-Pochhammer symbol* $(u;q)_n$ is defined so that

$$\frac{(u;q)_{n+1}}{(u;q)_n} = 1 - uq^n. \tag{3}$$

Namely,

$$(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1}).$$

The *basic hypergeometric series* is

$${}_r\phi_s(\boldsymbol{a};\boldsymbol{b};z;q) \coloneqq \sum_{n=0}^{\infty} \frac{(\boldsymbol{a};q)_n}{(q,\boldsymbol{b};q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n} z^n. \tag{4}$$

As $q \to 1$, since $\frac{(q^u;q)_n}{(1-q)^n} \to (u)_n$, we have

$${}_r\phi_s(q^{\boldsymbol{a}};q^{\boldsymbol{b}};(1-q)^{r-s-1}z;q) \to {}_rf_s(\boldsymbol{a};\boldsymbol{b};z).$$

There is a similar q-difference equation characterization of ${}_r\phi_s$.

Basic hypergeometric series

In 1846, Heine introduced the following q-deformation:
The *q-Pochhammer symbol* $(u;q)_n$ is defined so that

$$\frac{(u;q)_{n+1}}{(u;q)_n} = 1 - uq^n. \tag{3}$$

Namely,

$$(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1}).$$

The *basic hypergeometric series* is

$${}_r\phi_s(\boldsymbol{a};\boldsymbol{b};z;q) \coloneqq \sum_{n=0}^{\infty} \frac{(\boldsymbol{a};q)_n}{(q,\boldsymbol{b};q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} z^n. \tag{4}$$

As $q \to 1$, since $\frac{(q^u;q)_n}{(1-q)^n} \to (u)_n$, we have

$${}_r\phi_s(q^{\boldsymbol{a}};q^{\boldsymbol{b}};(1-q)^{r-s-1}z;q) \to {}_rf_s(\boldsymbol{a};\boldsymbol{b};z).$$

There is a similar q-difference equation characterization of ${}_r\phi_s$.

Basic hypergeometric series

In 1846, Heine introduced the following q-deformation:
The *q-Pochhammer symbol* $(u;q)_n$ is defined so that

$$\frac{(u;q)_{n+1}}{(u;q)_n} = 1 - uq^n. \tag{3}$$

Namely,

$$(u;q)_n = (1-u)(1-uq)\cdots(1-uq^{n-1}).$$

The *basic hypergeometric series* is

$${}_r\phi_s(\boldsymbol{a};\boldsymbol{b};z;q) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{a};q)_n}{(q,\boldsymbol{b};q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} z^n. \tag{4}$$

As $q \to 1$, since $\frac{(q^u;q)_n}{(1-q)^n} \to (u)_n$, we have

$${}_r\phi_s(q^{\boldsymbol{a}};q^{\boldsymbol{b}};(1-q)^{r-s-1}z;q) \to {}_rf_s(\boldsymbol{a};\boldsymbol{b};z).$$

There is a similar q-difference equation characterization of ${}_r\phi_s$.

Zonal hypergeometric series

In order to describe spherical functions and invariant integrals on symmetric spaces, especially for multivariate statistical distributions, Herz (1955) introduced *hypergeometric functions with matrix argument* via (multivariate) Laplace transform.

Later, Constantine (1963) realized Herz's function can be given by a series expansion involving *zonal polynomials*:

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{X}) = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \frac{C_\lambda(\boldsymbol{X})}{|\lambda|!}, $$

where λ runs over partitions of length at most n,

$$ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0, $$

$|\lambda| = \lambda_1 + \cdots + \lambda_n$, X is an n-by-n symmetric positive-definite matrix, $(u)_\lambda$ is a generalization of usual Pochhammer symbol, and C_λ is the *zonal polynomial*. Zonal polynomials depend only on the eigenvalues $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $\boldsymbol{X}$, and $C_\lambda(\boldsymbol{x})$ is a homogeneous symmetric polynomial of degree $|\lambda|$.

Zonal hypergeometric series

In order to describe spherical functions and invariant integrals on symmetric spaces, especially for multivariate statistical distributions, Herz (1955) introduced *hypergeometric functions with matrix argument* via (multivariate) Laplace transform.

Later, Constantine (1963) realized Herz's function can be given by a series expansion involving *zonal polynomials*:

$$ {}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{X}) = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \frac{C_\lambda(\boldsymbol{X})}{|\lambda|!}, $$

where λ runs over partitions of length at most n,

$$ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0, $$

$|\lambda| = \lambda_1 + \cdots + \lambda_n$, X is an n-by-n symmetric positive-definite matrix, $(u)_\lambda$ is a generalization of usual Pochhammer symbol, and C_λ is the *zonal polynomial*.

Zonal polynomials depend only on the eigenvalues $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $\boldsymbol{X}$, and $C_\lambda(\boldsymbol{x})$ is a homogeneous symmetric polynomial of degree $|\lambda|$.

Zonal hypergeometric series

In order to describe spherical functions and invariant integrals on symmetric spaces, especially for multivariate statistical distributions, Herz (1955) introduced *hypergeometric functions with matrix argument* via (multivariate) Laplace transform.
Later, Constantine (1963) realized Herz's function can be given by a series expansion involving *zonal polynomials*:

$$ {}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{X}) = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \frac{C_\lambda(\boldsymbol{X})}{|\lambda|!}, $$

where λ runs over partitions of length at most n,

$$ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0, $$

$|\lambda| = \lambda_1 + \cdots + \lambda_n$, X is an n-by-n symmetric positive-definite matrix, $(u)_\lambda$ is a generalization of usual Pochhammer symbol, and C_λ is the *zonal polynomial*.
Zonal polynomials depend only on the eigenvalues $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $\boldsymbol{X}$, and $C_\lambda(\boldsymbol{x})$ is a homogeneous symmetric polynomial of degree $|\lambda|$.

Jack and Macdonald hypergeometric series

Around 1970, Jack introduced *Jack polynomials* $J_\lambda(\alpha)$, unifying Schur ($\alpha = 1$) and zonal ($\alpha = 2$) polynomials.
In the 1980s, Macdonald introduced *Macdonald polynomials* $J_\lambda(q,t)$, unifying Jack ($t = q^{1/\alpha}$, $q \to 1$) and Hall–Littlewood ($t = 0$) polynomials.

Macdonald also introduced further generalizations of hypergeometric series, involving *Jack and Macdonald polynomials*

$$ {}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha) = \sum_\lambda \frac{(\boldsymbol{a};\alpha)_\lambda}{(\boldsymbol{b};\alpha)_\lambda} J_\lambda^\#(\boldsymbol{x};\alpha), \tag{5} $$

$$ {}_r\Phi_s(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};q,t) = \sum_\lambda \frac{(\boldsymbol{a};q,t)_\lambda}{(\boldsymbol{b};q,t)_\lambda} J_\lambda^\#(\boldsymbol{x};q,t), \tag{6} $$

and posed many fundamental questions.
Zonal hypergeometric series = Jack hypergeometric series with $\alpha = 2$.

There are two-alphabet versions: ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x},\boldsymbol{y};\alpha)$ and ${}_r\Phi_s(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x},\boldsymbol{y};q,t)$.

Jack and Macdonald hypergeometric series

Around 1970, Jack introduced *Jack polynomials* $J_\lambda(\alpha)$, unifying Schur ($\alpha = 1$) and zonal ($\alpha = 2$) polynomials.
In the 1980s, Macdonald introduced *Macdonald polynomials* $J_\lambda(q,t)$, unifying Jack ($t = q^{1/\alpha}, q \to 1$) and Hall–Littlewood ($t = 0$) polynomials.

Macdonald also introduced further generalizations of hypergeometric series, involving *Jack and Macdonald polynomials*

$$ {}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha) = \sum_\lambda \frac{(\boldsymbol{a};\alpha)_\lambda}{(\boldsymbol{b};\alpha)_\lambda} J^{\#}_\lambda(\boldsymbol{x};\alpha), \tag{5} $$

$$ {}_r\Phi_s(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};q,t) = \sum_\lambda \frac{(\boldsymbol{a};q,t)_\lambda}{(\boldsymbol{b};q,t)_\lambda} J^{\#}_\lambda(\boldsymbol{x};q,t), \tag{6} $$

and posed many fundamental questions.
Zonal hypergeometric series = Jack hypergeometric series with $\alpha = 2$.

There are two-alphabet versions: ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x},\boldsymbol{y};\alpha)$ and ${}_r\Phi_s(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x},\boldsymbol{y};q,t)$.

Jack and Macdonald hypergeometric series

Around 1970, Jack introduced *Jack polynomials* $J_\lambda(\alpha)$, unifying Schur ($\alpha = 1$) and zonal ($\alpha = 2$) polynomials.
In the 1980s, Macdonald introduced *Macdonald polynomials* $J_\lambda(q, t)$, unifying Jack ($t = q^{1/\alpha}, q \to 1$) and Hall–Littlewood ($t = 0$) polynomials.

Macdonald also introduced further generalizations of hypergeometric series, involving *Jack and Macdonald polynomials*

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha) = \sum_\lambda \frac{(\boldsymbol{a}; \alpha)_\lambda}{(\boldsymbol{b}; \alpha)_\lambda} J_\lambda^{\#}(\boldsymbol{x}; \alpha), \tag{5} $$

$$ {}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; q, t) = \sum_\lambda \frac{(\boldsymbol{a}; q, t)_\lambda}{(\boldsymbol{b}; q, t)_\lambda} J_\lambda^{\#}(\boldsymbol{x}; q, t), \tag{6} $$

and posed many fundamental questions.
Zonal hypergeometric series = Jack hypergeometric series with $\alpha = 2$.

There are two-alphabet versions: ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; \alpha)$ and ${}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; q, t)$.

Jack and Macdonald hypergeometric series

Around 1970, Jack introduced *Jack polynomials* $J_\lambda(\alpha)$, unifying Schur ($\alpha = 1$) and zonal ($\alpha = 2$) polynomials.
In the 1980s, Macdonald introduced *Macdonald polynomials* $J_\lambda(q, t)$, unifying Jack ($t = q^{1/\alpha}, q \to 1$) and Hall–Littlewood ($t = 0$) polynomials.

Macdonald also introduced further generalizations of hypergeometric series, involving *Jack and Macdonald polynomials*

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha) = \sum_\lambda \frac{(\boldsymbol{a}; \alpha)_\lambda}{(\boldsymbol{b}; \alpha)_\lambda} J^{\#}_\lambda(\boldsymbol{x}; \alpha), \tag{5} $$

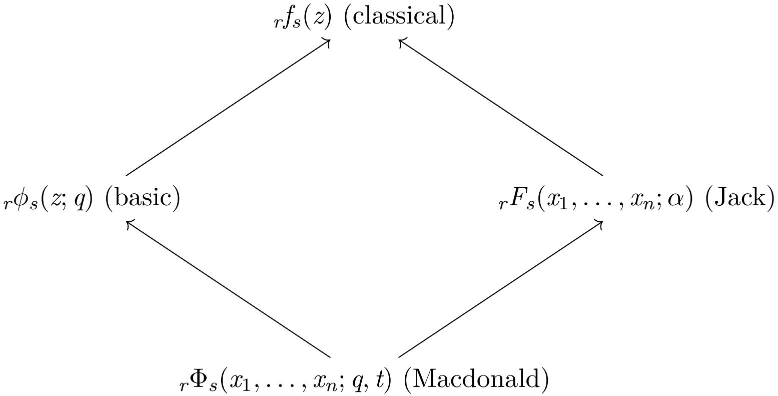
$$ {}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; q, t) = \sum_\lambda \frac{(\boldsymbol{a}; q, t)_\lambda}{(\boldsymbol{b}; q, t)_\lambda} J^{\#}_\lambda(\boldsymbol{x}; q, t), \tag{6} $$

and posed many fundamental questions.
Zonal hypergeometric series = Jack hypergeometric series with $\alpha = 2$.

There are two-alphabet versions: ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; \alpha)$ and ${}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; q, t)$.

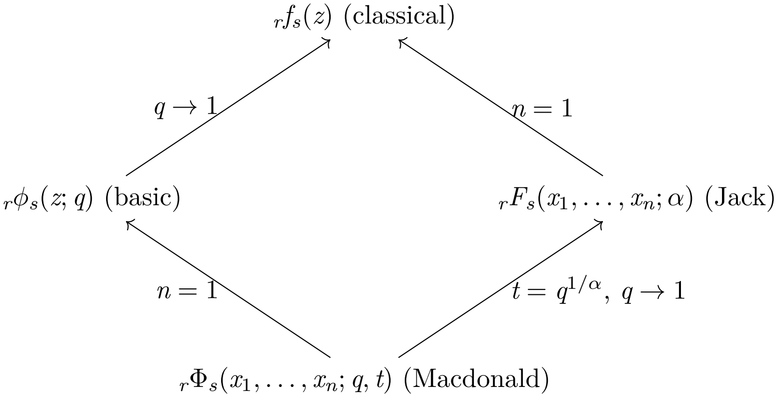
The relations among the hypergeometric series

So far, we have seen four families of hypergeometric series: the classical, the basic, the Jack and the Macdonald hypergeometric series



The relations among the hypergeometric series

So far, we have seen four families of hypergeometric series: the classical, the basic, the Jack and the Macdonald hypergeometric series



Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Known results

- **Question**: Differential equations for zonal/Jack hypergeometric series?

  Answer: Zonal: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.

  Jack: [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(\boldsymbol{x})$ and [Baker–Forrester, '97] solved ${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$.

- **Question**: Macdonald?

  Answer: [Kaneko, '96] solved ${}_2\Phi_1(\boldsymbol{x})$.

- **Our work**: we solve everything.

Our work

Jack case

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{7}$$

subject to a stability condition.
This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

Our work

Jack case

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{7}$$

subject to a stability condition.

This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

Our work

Jack case

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\boldsymbol{0}) = 1, \tag{7}$$

subject to a stability condition.
This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\boldsymbol{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\boldsymbol{0}, \boldsymbol{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

Our work

Jack case

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{7}$$

subject to a stability condition.
This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{7}$$

subject to a stability condition.
This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_\lambda C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_\lambda C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

We find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{7}$$

subject to a stability condition.
This Theorem generalizes all results on the previous slide.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{8}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

Pochhammer symbols

The Pochhammer symbol $(u)_n = u(u+1)\cdots(u+n-1)$ can be represented by the tableau

0	1	$\cdots$	$n-1$	n

so that

$$\frac{(u)_{n+1}}{(u)_n} = u + n.$$

For partitions, the *α-Pochhammer symbol* is defined so that

$$\frac{(u;\alpha)_{\mu\cup(i,j)}}{(u;\alpha)_\mu} = u + \rho(i,j),$$

where $\rho(i,j) = j - 1 - \frac{i-1}{\alpha}$ is the α-content of (i,j).

For example, $\frac{(u;\alpha)_{(322)}}{(u;\alpha)_{(321)}} = u + \rho(3,2)$.

$\rho(1,1)$	$\rho(1,2)$	$\rho(1,3)$
$\rho(2,1)$	$\rho(2,2)$	
$\rho(3,1)$	$\rho(3,2)$	

From now on, write $(u)_\lambda = (u;\alpha)_\lambda$ and $(\boldsymbol{a})_\lambda = (a_1)_\lambda \cdots (a_p)_\lambda$.

Pochhammer symbols

The Pochhammer symbol $(u)_n = u(u+1)\cdots(u+n-1)$ can be represented by the tableau

0	1	$\cdots$	$n-1$	n

so that

$$\frac{(u)_{n+1}}{(u)_n} = u + n.$$

For partitions, the *α-Pochhammer symbol* is defined so that

$$\frac{(u;\alpha)_{\mu\cup(i,j)}}{(u;\alpha)_\mu} = u + \rho(i,j),$$

where $\rho(i,j) = j - 1 - \frac{i-1}{\alpha}$ is the α-content of (i,j).

For example, $\frac{(u;\alpha)_{(322)}}{(u;\alpha)_{(321)}} = u + \rho(3,2)$.

$\rho(1,1)$	$\rho(1,2)$	$\rho(1,3)$
$\rho(2,1)$	$\rho(2,2)$	
$\rho(3,1)$	$\rho(3,2)$	

From now on, write $(u)_\lambda = (u;\alpha)_\lambda$ and $(\boldsymbol{a})_\lambda = (a_1)_\lambda \cdots (a_p)_\lambda$.

Pochhammer symbols

The Pochhammer symbol $(u)_n = u(u+1)\cdots(u+n-1)$ can be represented by the tableau

$$\boxed{0}\boxed{1}\ \cdots\ \boxed{n-1}\ \color{blue}{n}$$

so that

$$\frac{(u)_{n+1}}{(u)_n} = u + \color{blue}{n}.$$

For partitions, the *α-Pochhammer symbol* is defined so that

$$\frac{(u;\alpha)_{\mu\cup(i,j)}}{(u;\alpha)_\mu} = u + \rho(i,j),$$

where $\rho(i,j) = j - 1 - \frac{i-1}{\alpha}$ is the α-content of (i,j).

For example, $\frac{(u;\alpha)_{(322)}}{(u;\alpha)_{(321)}} = u + \rho(3,2)$.

$\rho(1,1)$	$\rho(1,2)$	$\rho(1,3)$
$\rho(2,1)$	$\rho(2,2)$	
$\rho(3,1)$	$\rho(3,2)$	

From now on, write $(u)_\lambda = (u;\alpha)_\lambda$ and $(\boldsymbol{a})_\lambda = (a_1)_\lambda \cdots (a_p)_\lambda$.

Pochhammer symbols

Define the statistic

$$\rho(\lambda) = \sum_{(i,j)\in\lambda} \rho(i,j), \quad \rho(\lambda/\mu) = \rho(\lambda) - \rho(\mu).$$

We say λ *covers* μ, written as $\lambda :\supset \mu$, if $\lambda = \mu \cup (i,j)$. Then

$$\frac{(u)_\lambda}{(u)_\mu} = u + \rho(\lambda/\mu),$$

and for a tuple $\boldsymbol{a}$, we have

$$\frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} = \prod_{k=1}^{p} \big(a_k + \rho(\lambda/\mu)\big) = \sum_{r=0}^{p} e_{p-r}(\boldsymbol{a})\rho(\lambda/\mu)^r, \tag{10}$$

where e_k is the k-th elementary symmetric function.

$$e_k(\boldsymbol{a}) = \sum_{i_1<\cdots<i_k} a_{i_1}\dots a_{i_k}.$$

Pochhammer symbols

Define the statistic

$$\rho(\lambda) = \sum_{(i,j)\in\lambda} \rho(i,j), \quad \rho(\lambda/\mu) = \rho(\lambda) - \rho(\mu).$$

We say λ *covers* μ, written as $\lambda :\supset \mu$, if $\lambda = \mu \cup (i,j)$. Then

$$\frac{(u)_\lambda}{(u)_\mu} = u + \rho(\lambda/\mu),$$

and for a tuple $\boldsymbol{a}$, we have

$$\frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} = \prod_{k=1}^{p} \big(a_k + \rho(\lambda/\mu)\big) = \sum_{r=0}^{p} e_{p-r}(\boldsymbol{a})\rho(\lambda/\mu)^r, \tag{10}$$

where e_k is the k-th elementary symmetric function.

$$e_k(\boldsymbol{a}) = \sum_{i_1<\cdots<i_k} a_{i_1} \ldots a_{i_k}.$$

Jack polynomials

Jack polynomials $(J_\lambda^\#(x;\alpha))$ form a basis of the algebra of symmetric polynomials $\Lambda_\alpha = \mathbb{Q}(\alpha)[x_1,\ldots,x_n]^{S_n}$ over $\mathbb{Q}(\alpha)$.

On Λ_α, there is a naturally defined inner product, $\langle\cdot,\cdot\rangle_\alpha$.
There is a commuting family of differential operators, known as *Debiard–Sekiguchi operators*,

$$D(t) := \frac{1}{\prod_{i<j}(x_i - x_j)} \det\left(x_i^{n-j}\left(x_i\partial_i - (j-1)/\alpha + t\right)\right)_{1\leqslant i,j\leqslant n},$$

that is self-adjoint:

$$\langle D(t)u, v\rangle_\alpha = \langle u, D(t)v\rangle_\alpha, \quad u, v \in \Lambda_\alpha.$$

Jack polynomials appear as common eigenvectors of $D(t)$:

$$D(t)J_\lambda^\# = d_\lambda(t)J_\lambda^\#, \quad \forall\lambda \tag{11}$$

with eigenvalues given by

$$d_\lambda(t) = \prod_{i=1}^{n}(\lambda_i - (i-1)/\alpha + t).$$

Jack polynomials

Jack polynomials $(J_\lambda^\#(x;\alpha))$ form a basis of the algebra of symmetric polynomials $\Lambda_\alpha = \mathbb{Q}(\alpha)[x_1, \ldots, x_n]^{S_n}$ over $\mathbb{Q}(\alpha)$.

On Λ_α, there is a naturally defined inner product, $\langle \cdot, \cdot \rangle_\alpha$.
There is a commuting family of differential operators, known as *Debiard–Sekiguchi operators*,

$$D(t) := \frac{1}{\prod_{i<j}(x_i - x_j)} \det\left(x_i^{n-j}\left(x_i\partial_i - (j-1)/\alpha + t\right)\right)_{1 \leqslant i,j \leqslant n},$$

that is self-adjoint:

$$\langle D(t)u, v\rangle_\alpha = \langle u, D(t)v\rangle_\alpha, \quad u, v \in \Lambda_\alpha.$$

Jack polynomials appear as common eigenvectors of $D(t)$:

$$D(t)J_\lambda^\# = d_\lambda(t)J_\lambda^\#, \quad \forall\lambda \tag{11}$$

with eigenvalues given by

$$d_\lambda(t) = \prod_{i=1}^{n}(\lambda_i - (i-1)/\alpha + t).$$

Jack polynomials

Jack polynomials ($J^{\#}_{\lambda}(x;\alpha)$) form a basis of the algebra of symmetric polynomials $\Lambda_\alpha = \mathbb{Q}(\alpha)[x_1, \ldots, x_n]^{S_n}$ over $\mathbb{Q}(\alpha)$.

On Λ_α, there is a naturally defined inner product, $\langle \cdot, \cdot \rangle_\alpha$.
There is a commuting family of differential operators, known as *Debiard–Sekiguchi operators*,

$$D(t) := \frac{1}{\prod_{i<j}(x_i - x_j)} \det\left(x_i^{n-j} \left(x_i \partial_i - (j-1)/\alpha + t \right) \right)_{1 \leqslant i,j \leqslant n},$$

that is self-adjoint:

$$\langle D(t)u, v \rangle_\alpha = \langle u, D(t)v \rangle_\alpha, \quad u, v \in \Lambda_\alpha.$$

Jack polynomials appear as common eigenvectors of $D(t)$:

$$D(t) J^{\#}_{\lambda} = d_\lambda(t) J^{\#}_{\lambda}, \quad \forall \lambda \tag{11}$$

with eigenvalues given by

$$d_\lambda(t) = \prod_{i=1}^{n} (\lambda_i - (i-1)/\alpha + t).$$

Jack polynomials

Jack polynomials ($J^{\#}_\lambda(x;\alpha)$) form a basis of the algebra of symmetric polynomials $\Lambda_\alpha = \mathbb{Q}(\alpha)[x_1,\ldots,x_n]^{S_n}$ over $\mathbb{Q}(\alpha)$.

On Λ_α, there is a naturally defined inner product, $\langle\cdot,\cdot\rangle_\alpha$.
There is a commuting family of differential operators, known as *Debiard–Sekiguchi operators*,

$$D(t) := \frac{1}{\prod_{i<j}(x_i-x_j)} \det\left(x_i^{n-j}\left(x_i\partial_i-(j-1)/\alpha+t\right)\right)_{1\leqslant i,j\leqslant n},$$

that is self-adjoint:

$$\langle D(t)u,v\rangle_\alpha = \langle u, D(t)v\rangle_\alpha, \quad u,v\in\Lambda_\alpha.$$

Jack polynomials appear as common eigenvectors of $D(t)$:

$$D(t)J^{\#}_\lambda = d_\lambda(t)J^{\#}_\lambda, \quad \forall\lambda \tag{11}$$

with eigenvalues given by

$$d_\lambda(t) = \prod_{i=1}^{n}(\lambda_i-(i-1)/\alpha+t).$$

Pieri formula

By [Okounkov–Olshanski, '97], [Lassalle, '98], [Sahi, '11], we have

$$e_1 \cdot J^{\#}_{\mu} = \sum_{\lambda :\supset \mu} \binom{\lambda}{\mu} J^{\#}_{\lambda},$$

where

$$e_1 = x_1 + \cdots + x_n,$$

and $\binom{\lambda}{\mu}$ are generalized binomial coefficients.

Laplace–Beltrami operator

Consider the *Laplace–Beltrami operator* $\Box$:

$$\Box = \sum_{i=1}^{n} \frac{1}{2} x_i^2 \partial_i^2 + \frac{1}{\alpha} \sum_{1 \leqslant i \neq j \leqslant n} \frac{x_i x_j}{x_i - x_j} \partial_i.$$

It acts diagonally on $(J_\lambda^\#)$ as

$$\Box(J_\lambda^\#) = \rho(\lambda) \cdot J_\lambda^\#.$$

It is related to the Debiard–Sekiguchi operator $D(t)$.

Adjoint action

Recall that $\mathrm{ad}_A(B) = [A, B] = AB - BA$, where A, B are operators.
Observe that

$$[\square, e_1](J^{\#}_{\mu}) = (\square e_1 - e_1 \square)(J^{\#}_{\mu}) = \sum_{\lambda :\supset \mu} \rho(\lambda) \binom{\lambda}{\mu} J^{\#}_{\lambda} - \rho(\mu) \sum_{\lambda :\supset \mu} \binom{\lambda}{\mu} J^{\#}_{\lambda}$$
$$= \sum_{\lambda :\supset \mu} \rho(\lambda/\mu) \binom{\lambda}{\mu} J^{\#}_{\lambda},$$

where $\rho(\lambda/\mu) = \rho(\lambda) - \rho(\mu)$.
More generally, repeated adjoint actions give

$$\left(\mathrm{ad}^{r}_{\square}(e_1)\right)\left(J^{\#}_{\mu}\right) = \sum_{\lambda :\supset \mu} \rho(\lambda/\mu)^{r} \binom{\lambda}{\mu} J^{\#}_{\lambda}, \quad r \geqslant 0.$$

Adjoint action

Recall that $\mathrm{ad}_A(B) = [A, B] = AB - BA$, where A, B are operators.
Observe that

$$[\square, e_1](J^\#_\mu) = (\square e_1 - e_1\square)(J^\#_\mu) = \sum_{\lambda :\supset \mu} \rho(\lambda) \binom{\lambda}{\mu} J^\#_\lambda - \rho(\mu) \sum_{\lambda :\supset \mu} \binom{\lambda}{\mu} J^\#_\lambda$$
$$= \sum_{\lambda :\supset \mu} \rho(\lambda/\mu) \binom{\lambda}{\mu} J^\#_\lambda,$$

where $\rho(\lambda/\mu) = \rho(\lambda) - \rho(\mu)$.
More generally, repeated adjoint actions give

$$\left(\mathrm{ad}^r_\square(e_1)\right)(J^\#_\mu) = \sum_{\lambda :\supset \mu} \rho(\lambda/\mu)^r \binom{\lambda}{\mu} J^\#_\lambda, \quad r \geqslant 0.$$

The raising operator $\mathcal{R}$

Define

$$\mathcal{R} := \sum_{r=0}^{p} e_{p-r}(\boldsymbol{a})\,\mathrm{ad}_{\square}^{r}(e_1) = \prod_{k=1}^{p}(a_k + \mathrm{ad}_{\square})(e_1), \tag{12}$$

then

$$\begin{aligned}
\mathcal{R}(J_\mu^{\#}) &= \sum_{\lambda \supset \mu} \left(\sum_{r=0}^{p} e_{p-r}(\boldsymbol{a}) \rho(\lambda/\mu)^r \right) \binom{\lambda}{\mu} J_\lambda^{\#} \\
&= \sum_{\lambda \supset \mu} \prod_k (a_k + \rho(\lambda/\mu)) \binom{\lambda}{\mu} J_\lambda^{\#} \\
&= \sum_{\lambda \supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J_\lambda^{\#}.
\end{aligned}$$

The raising operator $\mathcal{R}$

Define

$$\mathcal{R} := \sum_{r=0}^{p} e_{p-r}(\boldsymbol{a}) \operatorname{ad}_{\square}^{r}(e_1) = \prod_{k=1}^{p} (a_k + \operatorname{ad}_{\square})(e_1), \tag{12}$$

then

$$\begin{aligned}
\mathcal{R}(J_{\mu}^{\#}) &= \sum_{\lambda :\supset \mu} \left(\sum_{r=0}^{p} e_{p-r}(\boldsymbol{a}) \rho(\lambda/\mu)^{r} \right) \binom{\lambda}{\mu} J_{\lambda}^{\#} \\
&= \sum_{\lambda :\supset \mu} \prod_{k} (a_k + \rho(\lambda/\mu)) \binom{\lambda}{\mu} J_{\lambda}^{\#} \\
&= \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_{\lambda}}{(\boldsymbol{a})_{\mu}} \binom{\lambda}{\mu} J_{\lambda}^{\#}.
\end{aligned}$$

The diagonal operator $\mathcal{N}$

Since

$$ {}_pF_q = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} J^{\#}_\mu, \quad \mathcal{R}(J^{\#}_\mu) = \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J^{\#}_\lambda. $$

Then

$$ \mathcal{R}({}_pF_q) = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J^{\#}_\lambda = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \cdot \underbrace{\sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu}}_{N(\lambda)} \cdot J^{\#}_\lambda = \mathcal{N}({}_pF_q). $$

Hence, it suffices to find a diagonal operator $\mathcal{N}$ such that

$$ \mathcal{N}(J^{\#}_\lambda) = N(\lambda) \cdot J^{\#}_\lambda. $$

The diagonal operator $\mathcal{N}$

Since

$$ {}_pF_q = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} J_\mu^{\#}, \quad \mathcal{R}(J_\mu^{\#}) = \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J_\lambda^{\#}. $$

Then

$$ \mathcal{R}({}_pF_q) = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J_\lambda^{\#} = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \cdot \underbrace{\sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu}}_{N(\lambda)} \cdot J_\lambda^{\#} = \mathcal{N}({}_pF_q). $$

Hence, it suffices to find a diagonal operator $\mathcal{N}$ such that

$$ \mathcal{N}(J_\lambda^{\#}) = N(\lambda) \cdot J_\lambda^{\#}. $$

The eigenvalues $N(\lambda)$ and $H_r(\lambda)$

$N(\lambda)$ can be decomposed into

$$N(\lambda) = \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) H_r(\lambda).$$

Proposition

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.

The eigenvalues $N(\lambda)$ and $H_r(\lambda)$

$N(\lambda)$ can be decomposed into

$$N(\lambda) = \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) H_r(\lambda).$$

Proposition

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.

The diagonal operator $\mathcal{N}$

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.

The Debiard–Sekiguchi operator $D(t)$ acts diagonally by $d_\lambda(t) = \prod_i (w_i + t)$. Define diagonal operators $\mathcal{H}_r$ by

$$\sum_{r=0}^{\infty} \mathcal{H}_r s^r := \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \frac{D(-1/s - 1 + 1/\alpha)}{D(-1/s - 1)}.$$

Then $\mathcal{H}_r(J_\lambda^\#) = H_r(\lambda) J_\lambda^\#$. Since $N(\lambda) = \sum_{r=0}^{q} e_{q-r}(b) H_r(\lambda)$.

$$\mathcal{N} := \sum_{r=0}^{q} e_{q-r}(b) \mathcal{H}_r, \quad \mathcal{N}(J_\lambda^\#) = N(\lambda) J_\lambda^\#.$$

The diagonal operator $\mathcal{N}$

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.
The Debiard–Sekiguchi operator $D(t)$ acts diagonally by $d_\lambda(t) = \prod_i (w_i + t)$.
Define diagonal operators $\mathcal{H}_r$ by

$$\sum_{r=0}^{\infty} \mathcal{H}_r s^r := \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \frac{D(-1/s - 1 + 1/\alpha)}{D(-1/s - 1)}.$$

Then $\mathcal{H}_r(J_\lambda^\#) = H_r(\lambda) J_\lambda^\#$. Since $N(\lambda) = \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) H_r(\lambda)$.

$$\mathcal{N} := \sum_{r=0}^{q} e_{q-r}(b) \mathcal{H}_r, \quad \mathcal{N}(J_\lambda^\#) = N(\lambda) J_\lambda^\#.$$

The diagonal operator $\mathcal{N}$

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha-1)s}{s^2} - \frac{\alpha + (\alpha+n-1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.
The Debiard–Sekiguchi operator $D(t)$ acts diagonally by $d_\lambda(t) = \prod_i (w_i + t)$.
Define diagonal operators $\mathcal{H}_r$ by

$$\sum_{r=0}^{\infty} \mathcal{H}_r s^r := \frac{\alpha + (\alpha-1)s}{s^2} - \frac{\alpha + (\alpha+n-1)s}{s^2} \frac{D(-1/s - 1 + 1/\alpha)}{D(-1/s-1)}.$$

Then $\mathcal{H}_r(J_\lambda^{\#}) = H_r(\lambda) J_\lambda^{\#}$. Since $N(\lambda) = \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) H_r(\lambda)$,

$$\mathcal{N} := \sum_{r=0}^{q} e_{q-r}(b)\mathcal{H}_r, \quad \mathcal{N}(J_\lambda^{\#}) = N(\lambda) J_\lambda^{\#}.$$

The diagonal operator $\mathcal{N}$

$$\sum_{r=0}^{\infty} H_r(\lambda) s^r = \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \prod_{i=1}^{n} \frac{w_i - 1/s - 1 + 1/\alpha}{w_i - 1/s - 1},$$

where $w_i = \lambda_i - (i-1)/\alpha$.
The Debiard–Sekiguchi operator $D(t)$ acts diagonally by $d_\lambda(t) = \prod_i (w_i + t)$.
Define diagonal operators $\mathcal{H}_r$ by

$$\sum_{r=0}^{\infty} \mathcal{H}_r s^r := \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \frac{D(-1/s - 1 + 1/\alpha)}{D(-1/s - 1)}.$$

Then $\mathcal{H}_r(J_\lambda^\#) = H_r(\lambda) J_\lambda^\#$. Since $N(\lambda) = \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) H_r(\lambda)$.

$$\mathcal{N} := \sum_{r=0}^{q} e_{q-r}(b) \mathcal{H}_r, \quad \mathcal{N}(J_\lambda^\#) = N(\lambda) J_\lambda^\#.$$

Theorem (Theorem D)

The series ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha)$ *is the unique solution in the form*

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J^{\#}_\lambda(\boldsymbol{x};\alpha). \tag{13}$$

of the equation

$$(\mathcal{R} - \mathcal{N})\,(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{14}$$

Proof.

Plug (13) in (14), we get a recursion

$$C_\lambda(\alpha) \sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu} = \sum_{\mu \subset: \lambda} C_\mu(\alpha) \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu},$$

$$C_\varnothing = 1,$$

which determines uniquely $C_\lambda(\alpha) = \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda}$. □

Theorem (Theorem D)

The series ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha)$ *is the unique solution in the form*

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda^{\#}(\boldsymbol{x};\alpha). \tag{13}$$

of the equation

$$(\mathcal{R} - \mathcal{N})\,(F(\boldsymbol{x})) = 0, \quad F(\boldsymbol{0}) = 1, \tag{14}$$

Proof.

Plug (13) in (14), we get a recursion

$$C_\lambda(\alpha) \sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu} = \sum_{\mu \subset: \lambda} C_\mu(\alpha) \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu},$$

$$C_\varnothing = 1,$$

which determines uniquely $C_\lambda(\alpha) = \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda}$. □

The lowering operator $\mathcal{L}$

By [Okounkov–Olshanski, '97], [Lassalle, '98], [Sahi, '11], we have

$$e_1 \cdot J^{\#}_{\mu} = \sum_{\lambda :\supset \mu} \binom{\lambda}{\mu} J^{\#}_{\lambda}, \quad E_1\left(\frac{J^{\#}_{\lambda}}{J^{\#}_{\lambda}(\mathbf{1}_n)}\right) = \sum_{\mu \subset: \lambda} \binom{\lambda}{\mu} \frac{J^{\#}_{\mu}}{J^{\#}_{\mu}(\mathbf{1}_n)},$$

where

$$e_1 = x_1 + \cdots + x_n, \quad E_1 = \partial_1 + \cdots + \partial_n,$$

Similarly define

$$\mathcal{L} := \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) \operatorname{ad}^{r}_{-\square}(E_1) = \prod_{k=1}^{q} (b_k + \operatorname{ad}_{-\square})(E_1),$$

then

$$\mathcal{L}\left(\frac{J^{\#}_{\lambda}}{J_{\lambda}(\mathbf{1}_n)}\right) = \sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_{\lambda}}{(\boldsymbol{b})_{\mu}} \binom{\lambda}{\mu} \frac{J^{\#}_{\mu}}{J^{\#}_{\mu}(\mathbf{1}_n)}.$$

The diagonal operator $\mathcal{M}$

Then

$$\mathcal{L}({}_pF_q) = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} \cdot \underbrace{\sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} \frac{J^{\#}_\lambda(\boldsymbol{1}_n)}{J^{\#}_\mu(\boldsymbol{1}_n)}}_{M_n(\mu)} \cdot J^{\#}_\mu = \mathcal{M}({}_pF_q),$$

where $\mathcal{M}$ is a diagonal operator such that

$$\mathcal{M}(J_\mu) = M_n(\mu) \cdot J_\mu.$$

Theorem (Theorem C)

The series ${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha)$ *is the unique solution in the form* (13) *of the equation*

$$(\mathcal{L} - \mathcal{M})\,(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{15}$$

subject to an extra stability condition.

Sketch of proof.

Plug (13) in (15), we get an under-determined system of equations:

$$\sum_{\lambda:\supset\mu} C_\lambda(\alpha)\frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu}\binom{\lambda}{\mu}\frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)} = C_\mu(\alpha)\sum_{\lambda:\supset\mu}\frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu}\binom{\lambda}{\mu}\frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)},$$

$$C_\varnothing = 1.$$

The extra stability condition “doubles” the number of equations. With some effort, we can show that this determines $(C_\lambda(\alpha))$ uniquely. □

Theorem (Theorem C)

The series ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha)$ *is the unique solution in the form* (13) *of the equation*

$$(\mathcal{L} - \mathcal{M})\,(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{15}$$

subject to an extra stability condition.

Sketch of proof.

Plug (13) in (15), we get an under-determined system of equations:

$$\sum_{\lambda :\supset \mu} C_\lambda(\alpha) \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu} \frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)} = C_\mu(\alpha) \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} \frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)},$$

$$C_\varnothing = 1.$$

The extra stability condition "doubles" the number of equations. With some effort, we can show that this determines $(C_\lambda(\alpha))$ uniquely. □

Theorem (Theorem C)

The series ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha)$ *is the unique solution in the form* (13) *of the equation*

$$(\mathcal{L} - \mathcal{M})\,(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{15}$$

subject to an extra stability condition.

Sketch of proof.

Plug (13) in (15), we get an under-determined system of equations:

$$\sum_{\lambda :\supset \mu} C_\lambda(\alpha) \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu} \frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)} = C_\mu(\alpha) \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} \frac{J^\#_\lambda(\mathbf{1}_n)}{J^\#_\mu(\mathbf{1}_n)},$$

$$C_\varnothing = 1.$$

The extra stability condition “doubles” the number of equations. With some effort, we can show that this determines $(C_\lambda(\alpha))$ uniquely. □

Summary

In the Jack case, we find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{16}$$

subject to a stability condition.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{17}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{18}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

We construct analogous q-difference operators for Macdonald polynomials, and prove analogous characterization theorems.

Summary

In the Jack case, we find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$ and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, such that

Theorem C ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{M}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1, \tag{16}$$

subject to a stability condition.

Theorem D ${}_pF_q(\boldsymbol{x})$ is the unique solution of

$$\left(\mathcal{R}^{(\boldsymbol{x})} - \mathcal{N}^{(\boldsymbol{x})}\right)(F(\boldsymbol{x})) = 0, \quad F(\mathbf{0}) = 1. \tag{17}$$

Theorem A ${}_pF_q(\boldsymbol{x}, \boldsymbol{y})$ is the unique solution of

$$\left(\mathcal{L}^{(\boldsymbol{x})} - \mathcal{R}^{(\boldsymbol{y})}\right)(G(\boldsymbol{x}, \boldsymbol{y})) = 0, \quad G(\mathbf{0}, \mathbf{0}) = 1. \tag{18}$$

Here, we assume that $F(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{y})$ are in the form

$$F(\boldsymbol{x}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha), \quad G(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} C_\lambda(\alpha) J_\lambda(\boldsymbol{x}; \alpha) J_\lambda(\boldsymbol{y}; \alpha).$$

We construct analogous q-difference operators for Macdonald polynomials, and prove analogous characterization theorems.

Thank you!

Examples

Recall that the binomial formula and the Cauchy identity can be written as

$$F(x) = {}_1F_0(a; x; \alpha) = \sum_{\lambda} (a)_{\lambda} \frac{J_{\lambda}(x)}{j_{\lambda}} = \prod_{i=1}^{n} (1 - x_i)^{-a},$$

$$G(x, y) = {}_1F_0(n/\alpha; x, y; \alpha) = \sum_{\lambda} \frac{J_{\lambda}(x) J_{\lambda}(y)}{j_{\lambda}} = \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1/\alpha}.$$

In this case, the operators are

$$\mathcal{L} = \sum_i \partial_i, \quad \mathcal{M} = \sum_i (x_i \partial_i + a), \quad \mathcal{N} = \sum_i x_i \partial_i, \quad \mathcal{R} = \sum_i x_i (x_i \partial_i + a),$$

and the theorems read

$$\mathcal{L}(F) = F \cdot a \sum_i \frac{1}{1 - x_i} = \mathcal{M}(F), \quad \mathcal{N}(F) = F \cdot a \sum_i \frac{x_i}{1 - x_i} = \mathcal{R}(F),$$

$$\mathcal{L}^{(x)}(G) = G \cdot \frac{1}{\alpha} \sum_{i,j} \frac{y_j}{1 - x_i y_j} = \mathcal{R}^{(y)}(G).$$

Some results of Euler and Gauss

Euler's integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt,$$

for $\Re(c) > \Re(b) > 0$.

Gauss' summation formula

Assuming that $c \neq 0, -1, -2, \ldots$, Gauss proved that the series converges absolutely for $|z| < 1$ and gave the famous summation formula, for $\Re(c-a-b) > 0$,

$$F(a,b;c;z=1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$