Differential and q-Difference Equations Characterizing Hypergeometric Series Associated with Orthogonal Polynomials

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Partitions and symmetric polynomials

Fix $n \geqslant 1$, the number of variables.
A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, such that

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

Denote by $\mathcal{P}_n$ the set of such partitions. The *length* $\ell(\lambda)$ is the number of non-zero parts, and the *size* is $|\lambda| = \sum \lambda_i$.

Let $\Lambda_n = \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ be the algebra of *symmetric polynomials* in n variables.
Many (any) interesting bases of Λ_n are indexed by partitions in $\mathcal{P}_n$.

- the monomial $m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$
- the power sum $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, $p_r = m_{(r)} = \sum_i x_i^r$
- the elementary $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, $e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$

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Schur polynomials

On Λ_n, there is an inner product $\langle\cdot,\cdot\rangle$, called *Hall inner product*

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda,$$

where $z_\lambda = \prod_i i^{m_i} \cdot m_i!$, and m_i is the multiplicity of i in λ.
Then Schur polynomials (s_λ) is the unique family of symmetric polynomials satisfying:

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu, \quad \langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}.$$

Other definitions:

$$s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})} = \det(e_{\lambda'_i-i+j}) = \sum_{T\in \mathrm{SSYT}(\lambda)} \prod_{s\in\lambda} x_{T(s)} = \cdots$$

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Jack polynomials and Macdonald polynomials

Enlarge the field: $\Lambda_{n,\alpha} = \Lambda_n \otimes \mathbb{Q}(\alpha)$ and $\Lambda_{n,q,t} = \Lambda_n \otimes \mathbb{Q}(q,t)$.

On $\Lambda_{n,\alpha}$ and $\Lambda_{n,q,t}$, there are natural generalizations of the Hall inner product,

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}, \quad \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Then *Jack polynomials* ($P_\lambda(\alpha)$) and *Macdonald polynomials* ($P_\lambda(q,t)$) are defined as the unique family satisfying:

$$P_\lambda(\theta) = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu}(\theta) m_\mu,$$

$$\langle P_\lambda, P_\mu \rangle_\theta = 0, \quad \lambda \neq \mu,$$

where $\theta = \alpha$ or (q,t). This form is called *monic*. Schur polynomials can be obtained by $\alpha = 1$ or $q = t$.

They are no longer orthonormal:

$$\langle P_\lambda, P_\lambda \rangle_\theta = c'_\lambda / c_\lambda,$$

and define the *integral form* $J_\lambda = c_\lambda P_\lambda$ and its *dual form* $J^*_\lambda = P_\lambda / c'_\lambda$.

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Hypergeometric series

In 1955, in order to study of non-central Wishart distributions in statistics (and other topics), Herz introduced hypergeometric function ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{X})$ with symmetric matrix $\boldsymbol{X}$ as argument via Laplace transform.

Later in 1963, Constantine realized that Herz's function can be expanded into a series involving *zonal polynomials*, which can be obtained from Jack polynomials by setting $\alpha = 2$. They only depend on the eigenvalues $\boldsymbol{x} = (x_1, \dots, x_n)$ of $\boldsymbol{X}$.

Around 1988, Macdonald introduced further generalizations of *hypergeometric series associated with Jack and Macdonald polynomials*.

$$ {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; \alpha) = \sum_{\lambda} \frac{(\boldsymbol{a}; \alpha)_\lambda}{(\boldsymbol{b}; \alpha)_\lambda} J_\lambda^{\#}(\boldsymbol{x}; \alpha), \tag{1} $$

$$ {}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}; q, t) = \sum_{\lambda} \frac{(\boldsymbol{a}; q, t)_\lambda}{(\boldsymbol{b}; q, t)_\lambda} J_\lambda^{\#}(\boldsymbol{x}; q, t), \tag{2} $$

here $(\cdot; \theta)_\lambda$ are generalized Pochhammer symbols and $J_\lambda^{\#}(\boldsymbol{x}; \alpha) = \alpha^{|\lambda|} J_\lambda^{*}(\boldsymbol{x}; \alpha)$ and $J_\lambda^{\#}(\boldsymbol{x}; q, t) = t^{n(\lambda)} J_\lambda^{*}(\boldsymbol{x}; q, t)$. There are two-alphabet versions: ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; \alpha)$ and ${}_r\Phi_s(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{x}, \boldsymbol{y}; q, t)$.

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The relations among the hypergeometric series

$$ {}_pf_q(\boldsymbol{a}; \boldsymbol{b}; z) := \sum_{k=0}^{\infty} \frac{(\boldsymbol{a})_k}{(1, \boldsymbol{b})_k} z^k, \tag{3} $$

$$ {}_r\phi_s(\boldsymbol{a}; \boldsymbol{b}; z; q) := \sum_{k=0}^{\infty} \left((-1)^k q^{\binom{k}{2}} \right)^{s+1-r} \frac{(\boldsymbol{a}; q)_k}{(q, \boldsymbol{b}; q)_k} z^k. \tag{4} $$

${}_rf_s(z)$ (classical)

${}_r\phi_s(z; q)$ (basic) $\nearrow$ ${}_rf_s(z)$; ${}_rF_s(x_1, \ldots, x_n; \alpha)$ (Jack) $\nwarrow$ ${}_rf_s(z)$

${}_r\Phi_s(x_1, \ldots, x_n; q, t)$ (Macdonald) $\to$ ${}_r\phi_s(z; q)$ and ${}_rF_s(x_1, \ldots, x_n; \alpha)$

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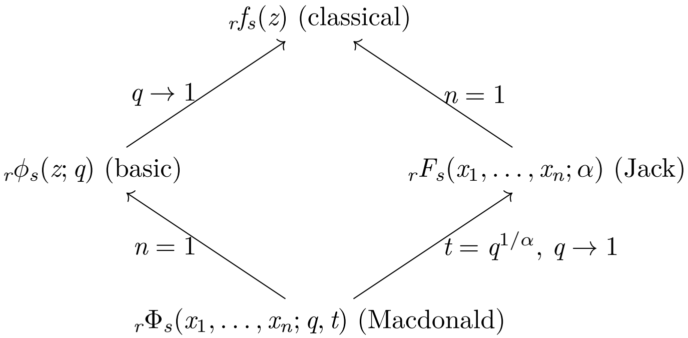

Differential and q-difference equations

- ${}_pf_q(z)$ is the unique formal power series solution of the equation

$$z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{j=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_j-1\right)(F)=z\prod_{i=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_i\right)(F),\quad F(0)=1. \tag{5}$$

- ${}_r\phi_s(z;q)$ is the unique formal power series solution of the equation

$$\Delta_1\Delta_{b_1/q}\cdots\Delta_{b_s/q}(F(z))=z\Delta_{a_1}\cdots\Delta_{a_r}(F(q^{s+1-r}z)),\quad F(0)=1, \tag{6}$$

where $\Delta_u(F(z))=uF(qz)-F(z)$.

See the Bateman Manuscript Project, [Andrews–Askey–Roy, Special functions], [Gasper–Rahman, Basic hypergeometric series].

Differential and q-difference equations

• Zonal
${}_2F_1$ (and degenerate cases): [Muirhead, '70] and [Constantine–Muirhead, '72]
${}_3F_2$ and ${}_2F_2$: [Fujikoshi, '75]

• Jack
${}_2F_1(\boldsymbol{x})$: [Yan, '92] and [Kaneko, '93], independently
${}_2F_1(\boldsymbol{x}, \boldsymbol{y})$: [Baker–Forrester, '97]

• Macdonald
${}_2\Phi_1(\boldsymbol{x})$: [Kaneko, '96]

Unlike univariate hypergeometric series, a systematic characterization of multivariate hypergeometric series has been lacking.

Our work: we find differential and q-difference equations characterizing ${}_pF_q$ and ${}_r\Phi_s$ for any p, q, r, s.

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Our work

For each of Jack and Macdonald cases, we find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$, and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, depending on the parameters $\boldsymbol{a}$, $\boldsymbol{b}$ and the number of variables n, such that

Theorem A Each of ${}_pF_q(\boldsymbol{x},\boldsymbol{y})$ and ${}_r\Phi_s(\boldsymbol{x},\boldsymbol{y})$ is the unique solution of

$$\mathcal{L}^{(\boldsymbol{x})}(G(\boldsymbol{x},\boldsymbol{y})) = \mathcal{R}^{(\boldsymbol{y})}(G(\boldsymbol{x},\boldsymbol{y})), \quad G(\mathbf{0},\mathbf{0}) = 1. \tag{7}$$

Theorem C Each of ${}_pF_q(\boldsymbol{x})$ and ${}_r\Phi_s(\boldsymbol{x})$ is the unique solution of

$$\mathcal{L}^{(\boldsymbol{x})}(F(\boldsymbol{x})) = \mathcal{M}^{(\boldsymbol{x})}(F(\boldsymbol{x})), \quad F(\mathbf{0}) = 1, \tag{8}$$

subject to a *stability condition* that ${}_pF_q(\boldsymbol{x}',\mathbf{0}')$ and ${}_r\Phi_s(\boldsymbol{x}',\mathbf{0}')$ satisfies $\mathcal{L}^{(\boldsymbol{x}')}(F(\boldsymbol{x}')) = \mathcal{M}^{(\boldsymbol{x}')}(F(\boldsymbol{x}')$, for $\boldsymbol{x}' = (x_1,\ldots,x_m)$, $m \leqslant n$. Theorems A and C generalize all previous results on multivariate series.

Theorem D Each of ${}_pF_q(\boldsymbol{x})$ and ${}_r\Phi_s(\boldsymbol{x})$ is the unique solution of

$$\mathcal{N}^{(\boldsymbol{x})}(F(\boldsymbol{x})) = \mathcal{R}^{(\boldsymbol{x})}(F(\boldsymbol{x})), \quad F(\mathbf{0}) = 1. \tag{9}$$

Here, we assume that $F(\boldsymbol{x}) = \sum_\lambda C_\lambda J_\lambda(\boldsymbol{x})$, $G(\boldsymbol{x},\boldsymbol{y}) = \sum_\lambda C_\lambda J_\lambda(\boldsymbol{x}) J_\lambda(\boldsymbol{y})$.

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Here, we assume that $F(\boldsymbol{x}) = \sum_\lambda C_\lambda J_\lambda(\boldsymbol{x})$, $G(\boldsymbol{x}, \boldsymbol{y}) = \sum_\lambda C_\lambda J_\lambda(\boldsymbol{x}) J_\lambda(\boldsymbol{y})$.

Our work

For each of Jack and Macdonald cases, we find a lowering operator $\mathcal{L}$, a raising operator $\mathcal{R}$, and two diagonal operators $\mathcal{M}$ and $\mathcal{N}$, depending on the parameters $\boldsymbol{a}$, $\boldsymbol{b}$ and the number of variables n, such that

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Theorem C Each of ${}_pF_q(\boldsymbol{x})$ and ${}_r\Phi_s(\boldsymbol{x})$ is the unique solution of

$$\mathcal{L}^{(\boldsymbol{x})}(F(\boldsymbol{x})) = \mathcal{M}^{(\boldsymbol{x})}(F(\boldsymbol{x})), \quad F(\mathbf{0}) = 1, \tag{8}$$

subject to a *stability condition* that ${}_pF_q(\boldsymbol{x}',\mathbf{0}')$ and ${}_r\Phi_s(\boldsymbol{x}',\mathbf{0}')$ satisfies $\mathcal{L}^{(\boldsymbol{x}')}(F(\boldsymbol{x}')) = \mathcal{M}^{(\boldsymbol{x}')}(F(\boldsymbol{x}')$, for $\boldsymbol{x}' = (x_1,\ldots,x_m)$, $m \leqslant n$. Theorems A and C generalize all previous results on multivariate series.

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Examples

Recall that the binomial formula and the Cauchy identity can be written as

$$F(x) = {}_1F_0(a; x; \alpha) = \sum_\lambda (a)_\lambda \frac{J_\lambda(x)}{j_\lambda} = \prod_{i=1}^n (1 - x_i)^{-a},$$

$$G(x, y) = {}_1F_0(n/\alpha; x, y; \alpha) = \sum_\lambda \frac{J_\lambda(x) J_\lambda(y)}{j_\lambda} = \prod_{i,j=1}^n (1 - x_i y_j)^{-1/\alpha}.$$

In this case, the operators are

$$\mathcal{L} = \sum_i \partial_i, \quad \mathcal{M} = \sum_i (x_i \partial_i + a), \quad \mathcal{N} = \sum_i x_i \partial_i, \quad \mathcal{R} = \sum_i x_i (x_i \partial_i + a),$$

and the theorems read

$$\mathcal{L}(F) = F \cdot a \sum_i \frac{1}{1 - x_i} = \mathcal{M}(F), \quad \mathcal{N}(F) = F \cdot a \sum_i \frac{x_i}{1 - x_i} = \mathcal{R}(F),$$

$$\mathcal{L}^{(x)}(G) = G \cdot \frac{1}{\alpha} \sum_{i,j} \frac{y_j}{1 - x_i y_j} = \mathcal{R}^{(y)}(G).$$

Lowering and raising operators

In the Jack case, the lowering operator $\mathcal{L}$ is built using the operator $E_1 = \sum \dfrac{\partial}{\partial x_i}$ and the Laplace–Beltrami operator

$$\square = \sum_{i=1}^{n} \frac{1}{2} x_i^2 \partial_i^2 + \frac{1}{\alpha} \sum_{1 \leqslant i \neq j \leqslant n} \frac{x_i x_j}{x_i - x_j} \partial_i.$$

We have

$$\mathcal{L} := \sum_{r=0}^{q} e_{q-r}(\boldsymbol{b}) \operatorname{ad}_{-\square}^{r}(E_1) = \prod_{k=1}^{q} (b_k + \operatorname{ad}_{-\square})(E_1).$$

The raising operator $\mathcal{R}$ uses $e_1 = \sum x_i$ and $\square$.

$$\mathcal{R} := \sum_{r=0}^{p} e_{p-r}(\boldsymbol{a}) \operatorname{ad}_{\square}^{r}(e_1) = \prod_{k=1}^{p} (a_k + \operatorname{ad}_{\square})(e_1).$$

Lowering and raising operators

$\mathcal{L}$ and $\mathcal{R}$ act nicely on Jack polynomials.

$$\mathcal{R}(J_\mu^\#) = \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J_\lambda^\#,$$

$$\mathcal{L}(\frac{J_\lambda}{J_\lambda(\mathbf{1}_n)}) = \sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu} \frac{J_\mu}{J_\mu(\mathbf{1}_n)}.$$

Diagonal operators

The diagonal operators $\mathcal{M}$ and $\mathcal{N}$ are defined according to their eigenvalues.

$$\mathcal{R}({}_pF_q) = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} \sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} J_\lambda^{\#} = \sum_{\lambda} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{b})_\lambda} \cdot \underbrace{\sum_{\mu \subset: \lambda} \frac{(\boldsymbol{b})_\lambda}{(\boldsymbol{b})_\mu} \binom{\lambda}{\mu}}_{N(\lambda)} \cdot J_\lambda^{\#} = \mathcal{N}({}_pF_q),$$

where

$$\mathcal{N}(J_\lambda) = N(\lambda) \cdot J_\lambda.$$

Similarly,

$$\mathcal{L}({}_pF_q) = \sum_{\mu} \frac{(\boldsymbol{a})_\mu}{(\boldsymbol{b})_\mu} \cdot \underbrace{\sum_{\lambda :\supset \mu} \frac{(\boldsymbol{a})_\lambda}{(\boldsymbol{a})_\mu} \binom{\lambda}{\mu} \frac{J_\lambda^{\#}(\mathbf{1}_n)}{J_\mu^{\#}(\mathbf{1}_n)}}_{M_n(\mu)} \cdot J_\mu^{\#} = \mathcal{M}({}_pF_q),$$

where

$$\mathcal{M}(J_\mu) = M_n(\mu) \cdot J_\mu.$$

Diagonal operators

Define diagonal operators $\mathcal{G}_{r,n}$ and $\mathcal{H}_r$ by

$$\sum_{r=0}^{\infty} \mathcal{G}_{r,n} s^r := \frac{\alpha}{s} \left(\frac{D(-1/s - 1/\alpha)}{D(-1/s)} - 1 \right),$$

$$\sum_{r=0}^{\infty} \mathcal{H}_r s^r := \frac{\alpha + (\alpha - 1)s}{s^2} - \frac{\alpha + (\alpha + n - 1)s}{s^2} \frac{D(-1/s - 1 + 1/\alpha)}{D(-1/s - 1)},$$

where $D(t)$ is the Debiard–Sekiguchi operator

$$D(t) := \frac{1}{\prod_{i<j}(x_i - x_j)} \det \left(x_i^{n-j} \left(x_i \partial_i - (j-1)/\alpha + t \right) \right)_{1 \leqslant i,j \leqslant n},$$

this operator is self-adjoint and acts diagonally on Jack polynomials:

$$D(t) J_\lambda^{\#} = d_\lambda(t) J_\lambda^{\#}, \quad \forall \lambda, \quad d_\lambda(t) = \prod_{i=1}^{n} (\lambda_i - (i-1)/\alpha + t). \tag{10}$$

Then define

$$\mathcal{M} := \sum_{r=0}^{p} e_{p-r}(\underline{a}) \mathcal{G}_{r,n}, \quad \mathcal{N} := \sum_{r=0}^{q} e_{q-r}(b) \mathcal{H}_r.$$

Thank you!