A Characterization of Macdonald's Jack Hypergeometric Series ${}_pF_q(\boldsymbol{x};\alpha)$ and ${}_pF_q(\boldsymbol{x},\boldsymbol{y};\alpha)$ via Differential Equations

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Abstract

Ian G. Macdonald introduced multivariable hypergeometric series ${}_pF_q(\boldsymbol{x};\alpha)$ and ${}_pF_q(\boldsymbol{x},\boldsymbol{y};\alpha)$ associated with Jack polynomials that generalize classical hypergeometric functions to the symmetric function setting. This work provides explicit partial differential equations that characterize the series as the unique formal power series solution of the equation. This resolves a long-standing question posed by Macdonald.

Univariate Hypergeometric Series

Hypergeometric functions are a cornerstone of special functions in mathematics and physics, providing solutions to a wide array of differential equations and unifying many classical functions within a generalized framework.

Euler (1769) and Gauss (1812) first studied the unique power series solution to the following second-order linear ODE

$$\left(z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-(a+b+1)z)\frac{\mathrm{d}}{\mathrm{d}z}-ab\right)(F(z))=0,\quad F(0)=1. \tag{1}$$

The series solution of it is known as the Gauss hypergeometric series

$${}_2F_1(a,b;c;z):=1+\frac{ab}{c}\frac{z}{1}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots. \tag{2}$$

In general, **hypergeometric series** is a natural generalization of ${}_2F_1$:

$${}_pF_q(\boldsymbol{a};\boldsymbol{b};z):=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{3}$$

where $\boldsymbol{a}=(a_1,\dots,a_p)$, $\boldsymbol{b}=(b_1,\dots,b_q)$ and $(c)_k:=c(c+1)\cdots(c+k-1)$ is the **Pochhammer symbol**. Similar to ${}_2F_1(z)$, ${}_pF_q(z)$ can be characterized by

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{k=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_k-1\right)-z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_k\right)\right)(F(z))=0,\quad F(0)=1. \tag{4}$$

Besides differential equations, central topics in the theory of hypergeometric functions include integral representations (Euler, Barnes, etc.), summation formulas (Gauss, Pfaff–Saalschütz, Dougall, etc.), asymptotics, and transformations (Euler, Pfaff, Kummer, etc.).

Multivariate Hypergeometric Series

Since the work of Herz (1955) and Constantine (1963), multivariate hypergeometric series associated with zonal polynomials have been studied for applications in representation theory, probability and statistics.

In 1970, Jack introduced Jack polynomials $J_\lambda(\boldsymbol{x};\alpha)$ as a unification of Schur polynomials ($\alpha=1$) and zonal polynomials ($\alpha=2$).

Around 1988, Macdonald introduced in his manuscript [arXiv:1309.4568] the following **Jack hypergeometric series**

$${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x};\alpha)=\sum_{\lambda}\frac{(a_1;\alpha)_k\cdots(a_p;\alpha)_k}{(b_1;\alpha)_k\cdots(b_q;\alpha)_k}\alpha^{|\lambda|}\frac{J_\lambda(\boldsymbol{x};\alpha)}{j_\lambda}, \tag{5}$$

$${}_pF_q(\boldsymbol{a};\boldsymbol{b};\boldsymbol{x},\boldsymbol{y};\alpha)=\sum_{\lambda}\frac{(a_1;\alpha)_k\cdots(a_p;\alpha)_k}{(b_1;\alpha)_k\cdots(b_q;\alpha)_k}\alpha^{|\lambda|}\frac{J_\lambda(\boldsymbol{x};\alpha)J_\lambda(\boldsymbol{y};\alpha)}{j_\lambda J_\lambda(\boldsymbol{1};\alpha)}. \tag{6}$$

Here, the sums run over partitions $\lambda=(\lambda_1,\dots,\lambda_n)$ with at most n parts, $(c;\alpha)_\lambda:=\prod_{(i,j)\in\lambda}\left(c+j-1-\frac{i-1}{\alpha}\right)$ is the α-**Pochhammer symbol**, J_λ is the integral Jack polynomial in the variables $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)$, and $\boldsymbol{1}=(1,\dots,1)$.

When $n=1$, the series ${}_pF_q(\boldsymbol{x};\alpha)$ reduces to the univariate hypergeometric series ${}_pF_q(x_1)$. It is then natural to ask if these multivariate series can be characterized by certain differential equations as well.

Goal

Find differential equations that characterize the Jack hypergeometric series ${}_pF_q(\boldsymbol{x};\alpha)$ and ${}_pF_q(\boldsymbol{x},\boldsymbol{y};\alpha)$.

There have been some partial results for small parameters: for $p\leqslant 3$ and $q\leqslant 2$ in the zonal case by Muirhead (1970), Constantine–Muirhead (1972), and Fujikoshi (1975); and for $p\leqslant 2$ and $q\leqslant 1$ in the Jack case by Macdonald (~1988), Yan (1992), Kaneko (1993), and Baker–Forrester (1997).

We resolve this question in full generality.

Our Work

Let $\mathscr{F}^{(\boldsymbol{x})}=\left\{\sum_\lambda C_\lambda J_\lambda(\boldsymbol{x})\right\}$ and $\mathscr{F}^{(\boldsymbol{x},\boldsymbol{y})}=\left\{\sum_\lambda C_\lambda J_\lambda(\boldsymbol{x})J_\lambda(\boldsymbol{y})\right\}$ be spaces of formal power series, and let $\boldsymbol{0}=(0,\dots,0)$. We find differential operators $\mathcal{A},\mathcal{B},\mathcal{C}$ (depending on p,q,n) such that the following hold for any p,q,n.

Theorem A The series ${}_pF_q(\boldsymbol{x},\boldsymbol{y};\alpha)$ is the unique solution in $\mathscr{F}^{(\boldsymbol{x},\boldsymbol{y})}$ of

$$\mathcal{A}(F(\boldsymbol{x},\boldsymbol{y}))=0,\quad F(\boldsymbol{0},\boldsymbol{0})=1. \tag{7}$$

Theorem B The series ${}_pF_q(\boldsymbol{x};\alpha)$ is the unique solution in $\mathscr{F}^{(\boldsymbol{x})}$ of

$$\mathcal{B}(F(\boldsymbol{x}))=0,\quad F(\boldsymbol{0})=1, \tag{8}$$

subject to a certain stability condition.

Theorem C The series ${}_pF_q(\boldsymbol{x};\alpha)$ is the unique solution in $\mathscr{F}^{(\boldsymbol{x})}$ of

$$\mathcal{C}(F(\boldsymbol{x}))=0,\quad F(\boldsymbol{0})=1. \tag{9}$$

The results mentioned above for small parameters can be recovered from our Theorems A and B. Our Theorem C is new even in those special cases.

Operators

Define the following operators:

$$E_1:=\sum_{i=1}^{n}\partial_i,\quad e_1:=\sum_{i=1}^{n}x_i,\quad D:=\frac{\det\left(x_i^{n-j}\left(x_i\partial_i-(j-1)/\alpha+t\right)\right)_{1\leqslant i,j\leqslant n}}{\det\left(x_i^{n-j}\right)_{1\leqslant i,j\leqslant n}},$$

where D is the **Debiard–Sekiguchi operator**. They act on $(J_\lambda(\boldsymbol{x}))$ by

$$E_1\left(\frac{J_\lambda(\boldsymbol{x})}{J_\lambda(\boldsymbol{1})}\right)=\sum_{\mu\subset\lambda}\binom{\lambda}{\mu}\frac{J_\mu(\boldsymbol{x})}{J_\mu(\boldsymbol{1})},\quad e_1\cdot\frac{J_\mu(\boldsymbol{x})}{j_\mu}=\alpha\sum_{\lambda\supset\mu}\binom{\lambda}{\mu}\frac{J_\lambda(\boldsymbol{x})}{j_\lambda},$$

$$D(J_\lambda(\boldsymbol{x}))=\prod_{i=1}^{n}\left(\lambda_i-(i-1)/\alpha+t\right)\cdot J_\lambda(\boldsymbol{x}).$$

Here, $\lambda:\supset\mu$ means $\lambda\supseteq\mu$ and $|\lambda|=|\mu|+1$.

One sees that the operator E_1 lowers the degree by one, e_1 raises it by one, while D preserves the degree, acting as an eigen-operator. Using these, we further construct differential operators $\mathcal{L},\mathcal{M},\mathcal{N},\mathcal{R}$, where $\mathcal{L}$ lowers the degree, $\mathcal{R}$ raises the degree, and $\mathcal{M}$ and $\mathcal{N}$ are eigen-operators. The operators $\mathcal{A},\mathcal{B},\mathcal{C}$ are then given by

$$\mathcal{A}=\mathcal{L}^{(\boldsymbol{x})}-\mathcal{R}^{(\boldsymbol{y})},\quad \mathcal{B}=\mathcal{L}-\mathcal{M},\quad \mathcal{C}=\mathcal{N}-\mathcal{R}. \tag{10}$$

An Example

For $p=1$, $q=0$, we have the binomial formula and the Cauchy identity:

$$F={}_1F_0(a;-;\boldsymbol{x};\alpha)=\sum_\lambda(a)_\lambda\frac{J_\lambda(\boldsymbol{x})}{j_\lambda}=\prod_{i=1}^{n}(1-x_i)^{-a},$$

$$G={}_1F_0(n/\alpha;-;\boldsymbol{x},\boldsymbol{y};\alpha)=\sum_\lambda\frac{J_\lambda(\boldsymbol{x})J_\lambda(\boldsymbol{y})}{j_\lambda}=\prod_{i,j=1}^{n}(1-x_iy_j)^{-1/\alpha}.$$

In this case, the operators are

$$\mathcal{L}=\sum_i\partial_i,\quad \mathcal{M}=\sum_i(x_i\partial_i+a),\quad \mathcal{N}=\sum_i x_i\partial_i,\quad \mathcal{R}=\sum_i x_i(x_i\partial_i+a),$$

and

$$\mathcal{L}(F)=F\cdot a\sum_i\frac{1}{1-x_i}=\mathcal{M}(F),$$

$$\mathcal{N}(F)=F\cdot a\sum_i\frac{x_i}{1-x_i}=\mathcal{R}(F),$$

$$\mathcal{L}^{(\boldsymbol{x})}(G)=G\cdot\frac{1}{\alpha}\sum_{i,j}\frac{y_j}{1-x_iy_j}=\mathcal{R}^{(\boldsymbol{y})}(G).$$