Partial orders on partitions & positivity on symmetric functions

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Joint work with Apoorva Khare and Siddhartha Sahi
arXiv: 2403.02490, 2509.19649
slides: https://sites.math.rutgers.edu/~hc813/

Today: fix $n \geqslant 1$ the number of variables, $x = (x_1, \ldots, x_n)$.

The classical AM–GM inequality asserts that

$$\frac{x_1 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 \cdots x_n}, \quad x \in [0, \infty)^n.$$

When $n = 2$, $\frac{x+y}{2} \geqslant \sqrt{xy}$, implicit in Euclid's *Elements* (c. 300 BC)
Substituting $x_i \mapsto x_i^n$, we have

$$\frac{x_1^n + \cdots + x_n^n}{n} \geqslant \frac{x_1 \cdots x_n}{1}, \quad x \in [0, \infty)^n. \tag{1}$$

The denominator = number of terms in the numerator = numerator evaluated at $\mathbf{1} = (1, \ldots, 1)$.
There are various generalizations of this classical inequality, see for example, [G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 1934.]

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Muirhead's inequality (1902) states that

$$\lambda \text{ dominates } \mu \iff \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})} \geqslant \frac{m_\mu(x)}{m_\mu(\mathbf{1})}, \quad x \in [0,\infty)^n. \tag{2}$$

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- $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ are **partitions**.

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

- We say λ **dominates** μ, denoted by $\lambda \succcurlyeq \mu$, if

$$\begin{gathered} \lambda_1 \geqslant \mu_1 \\ \lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2 \\ \vdots \\ \lambda_1 + \cdots + \lambda_{n-1} \geqslant \mu_1 + \cdots + \mu_{n-1} \\ |\lambda| = \lambda_1 + \cdots + \lambda_{n-1} + \lambda_n = \mu_1 + \cdots + \mu_{n-1} + \mu_n = |\mu| \end{gathered}$$

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- λ and μ are partitions
- m_λ is the **monomial symmetric polynomial**

$$m_\lambda(x) = \sum_\eta x^\eta, \quad x^\eta = x_1^{\eta_1} \cdots x_n^{\eta_n},$$

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In particular, the AM–GM inequality can be recovered: $(n) \succcurlyeq (1^n)$,

$$\frac{m_{(n)}(x)}{m_{(n)}(\mathbf{1})} = \frac{x_1^n + \cdots + x_n^n}{n} \geqslant \frac{x_1 \cdots x_n}{1} = \frac{m_{(1^n)}(x)}{m_{(1^n)}(\mathbf{1})}, \quad x \in [0,\infty)^n.$$

Newton's inequality (1707) states that $(\frac{e_k(x)}{e_k(\mathbf{1})})_k$ is log-concave:

$$\frac{e_k(x)}{e_k(\mathbf{1})}\frac{e_k(x)}{e_k(\mathbf{1})} \geqslant \frac{e_{k+1}(x)}{e_{k+1}(\mathbf{1})}\frac{e_{k-1}(x)}{e_{k-1}(\mathbf{1})}, \quad x \in [0,\infty)^n. \tag{3}$$

- e_k is the k-th **elementary symmetric polynomial**

$$e_k(x) = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}, \quad e_k(\mathbf{1}) = \binom{n}{k}.$$

Cuttler–Greene–Skandera (2011) generalized Newton's inequality into

$$\lambda \succcurlyeq \mu \iff \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} \geqslant \frac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})}, \quad x \in [0,\infty)^n. \tag{4}$$

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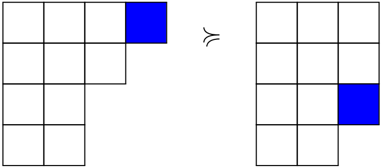
Young diagram:



$$\lambda = (4,3,2,2) \quad \succcurlyeq \quad \mu = (3,3,3,2)$$
$$\lambda' = (4,4,2,1) \quad \preccurlyeq \quad \mu' = (4,4,3)$$

For $|\lambda| = |\mu|$, we have $\lambda \succcurlyeq \mu \iff \mu' \succcurlyeq \lambda'$.

Cuttler–Greene–Skandera:

$$\lambda \succcurlyeq \mu \iff \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} \geqslant \frac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})}, \quad x \in [0, \infty)^n.$$

Let $\lambda' = (k, k) \preccurlyeq \mu' = (k+1, k-1)$, then Newton's inequality can be recovered:

$$\frac{e_k(x)}{e_k(\mathbf{1})} \frac{e_k(x)}{e_k(\mathbf{1})} \geqslant \frac{e_{k+1}(x)}{e_{k+1}(\mathbf{1})} \frac{e_{k-1}(x)}{e_{k-1}(\mathbf{1})}, \quad x \in [0, \infty)^n.$$

Gantmacher's inequality (1959) states that $(p_k(x))_k$ is log-convex:

$$p_{k+1}(x)p_{k-1}(x) \geqslant p_k(x)p_k(x), \quad x \in [0,\infty)^n. \tag{5}$$

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Cuttler–Greene–Skandera generalized Gantmacher's inequality into

$$\lambda \succcurlyeq \mu \iff \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})} \geqslant \frac{p_\mu(x)}{p_\mu(\mathbf{1})}, \quad x \in [0,\infty)^n. \tag{6}$$

- $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_n}$

Taking $\lambda = (k+1, k-1) \succcurlyeq \mu = (k,k)$ recovers Gantmacher's inequality

Schur proved $\left(\frac{h_k(x)}{h_k(\mathbf{1})}\right)_k$ is log-convex

$$\frac{h_{k+1}(x)}{h_{k+1}(\mathbf{1})}\frac{h_{k-1}(x)}{h_{k-1}(\mathbf{1})} \geqslant \frac{h_k(x)}{h_k(\mathbf{1})}\frac{h_k(x)}{h_k(\mathbf{1})}, \quad x \in [0,\infty)^n. \tag{7}$$

- h_k is the k-th complete homogeneous symmetric polynomial

$$h_k(x) = \sum_{i_1 \leqslant \cdots \leqslant i_k} x_{i_1}\cdots x_{i_k} = \sum_{|\lambda|=k} m_\lambda$$

Cuttler–Greene–Skandera proved

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What about $\Longleftarrow$?

For $|\lambda| = |\mu| \leqslant 7$, true.

For $|\lambda| = |\mu| \geqslant 8$, FALSE! See [Xu–Yao, arXiv:2505.08149].

For example $\lambda = (2^4) \not\succcurlyeq \mu = (3,1^5)$, but still $\frac{h_\lambda(x)}{h_\lambda(\mathbf{1})} \geqslant \frac{h_\mu(x)}{h_\mu(\mathbf{1})}$.

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Having seen

$$\boxed{\lambda \succcurlyeq \mu \iff b_\lambda(x) \geqslant b_\mu(x), \quad x \in [0,\infty)^n} \tag{*}$$

for the bases (b_λ) with $b_\lambda(x) = \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}, \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})}, \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})}$.
Cuttler–Greene–Skandera asked if this is true for Schur polynomials.

$$\lambda \succcurlyeq \mu \iff \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} \geqslant \frac{s_\mu(x)}{s_\mu(\mathbf{1})}, \quad x \in [0,\infty)^n, \tag{9}$$

Cuttler–Greene–Skandera proved $\Longleftarrow$ and conjectured $\Longrightarrow$.
Later, proved by Sra (2016), using Harish-Chandra–Itzykson–Zuber integral and AM–GM inequality.

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- Schur polynomial s_λ is defined as

$$\begin{aligned} s_\lambda(x) &= \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})} \\ &= \det(h_{\lambda_i-i+j}) = \det(e_{\lambda'_i-i+j}) \\ &= \text{character of } GL_n(\mathbb{C})\text{-modules} \\ &= \text{Frobenius characteristic of } S_d\text{-modules} \\ &= \text{spherical function for } (GL_n(\mathbb{C}), U_n) \\ &= \text{generating function of semi-standard Young tableaux} \\ &= \cdots \end{aligned}$$

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Take $b_\lambda(x) = \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}$, then we have

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Answer: Yes! This is the motivation of our first paper 2403.02490.

The binomial formula for Schur polynomials is

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where the expansion coefficients are called **generalized binomial coefficients**. They were first studied by Lascoux (1978) to compute Chern classes for exterior and symmetric squares of vector bundles. See also [Macdonald, p. 47, Example 10].

- We say λ **contains** μ, $\lambda \supseteq \mu$, if $\lambda_i \geqslant \mu_i$ for $i = 1, \dots, n$.

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If $\lambda \supseteq \mu$, then by the monotonicity, $\binom{\lambda}{\nu} - \binom{\mu}{\nu} \geqslant 0$ for any ν. Namely, the difference is **Schur positive**.
Conversely, if $\lambda \not\supseteq \mu$, then for $\nu = \mu$, we have $\binom{\lambda}{\nu} = 0$ and $\binom{\mu}{\nu} = 1$, and the difference is not Schur positive.

$$\lambda \supseteq \mu \iff \frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \text{ is Schur positive} \tag{11}$$

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Jack polynomials $P_\lambda(\alpha)$ were introduced by Jack in 1970s, as a unification of Schur polynomials and zonal polynomials (spherical function for $(GL_n(\mathbb{R}), O_n)$).

They can be defined as follows:

$$\langle P_\lambda(\alpha), P_\mu(\alpha)\rangle_\alpha = 0, \quad \lambda \neq \mu$$

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They depend on a parameter α:

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Write $\Omega_\lambda(x;\alpha) = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}$ and $\widetilde{\Omega}_\lambda(x;\alpha) = \Omega_\lambda(x+\mathbf{1};\alpha)$.
When $\alpha = 0, 1, \infty$, Jack specializes to elementary, Schur, monomial.
Ω and $\widetilde{\Omega}$ become: E and $\widetilde{E}$, S_λ and $\widetilde{S}_\lambda$, M and $\widetilde{M}$.

$$\widetilde{E}_{\ydiagram{2,1,1}} - \widetilde{E}_{\ydiagram{1,1}} = E_{\ydiagram{2,1,1}} + 2E_{\ydiagram{1,1,1}} + 2E_{\ydiagram{2,1}} + 4E_{\ydiagram{1,1}} + E_{\ydiagram{2}} + 2E_{\ydiagram{1}}$$

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holds for $b_\lambda = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}, \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})}, \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}, \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}$ and zonal.
Surprisingly, also holds for power sum: $b_\lambda(x) = \frac{p_\lambda(x+\mathbf{1})}{p_\lambda(\mathbf{1})}$. Proof by induction on length of λ.

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$$\widetilde{\Omega}_{(3,1)} - \widetilde{\Omega}_{(2)} = \Omega_{(3,1)} + \tfrac{2\alpha+2}{2\alpha+1}\Omega_{(3)} + \tfrac{6\alpha+2}{2\alpha+1}\Omega_{(2,1)} + \tfrac{2\alpha+4}{\alpha+1}\Omega_{(2)} + \tfrac{3\alpha+1}{\alpha+1}\Omega_{(1,1)} + 2\Omega_{(1)}$$

$$\boxed{\lambda \supseteq \mu \iff b_\lambda(x+\mathbf{1}) - b_\mu(x+\mathbf{1}) \text{ is } b\text{-positive}} \qquad (***)$$

holds for $b_\lambda = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}, \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})}, \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}, \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}$ and zonal.

Surprisingly, also holds for power sum: $b_\lambda(x) = \frac{p_\lambda(x+\mathbf{1})}{p_\lambda(\mathbf{1})}$. Proof by induction on length of λ.

Write $\Omega_\lambda(x;\alpha) = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}$ and $\widetilde{\Omega}_\lambda(x;\alpha) = \Omega_\lambda(x+\mathbf{1};\alpha)$.
When $\alpha = 0, 1, \infty$, Jack specializes to elementary, Schur, monomial.
Ω and $\widetilde{\Omega}$ become: E and $\widetilde{E}$, S_λ and $\widetilde{S}_\lambda$, M and $\widetilde{M}$.

$$\widetilde{E}_{(2,1,1)} - \widetilde{E}_{(1,1)} = E_{(2,1,1)} + 2E_{(1,1,1)} + 2E_{(2,1)} + 4E_{(1,1)} + E_{(2)} + 2E_{(1)}$$

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$$\boxed{\lambda \succcurlyeq \mu \iff b_\lambda(x) \geqslant b_\mu(x), \quad x \in [0,\infty)^n} \tag{*}$$

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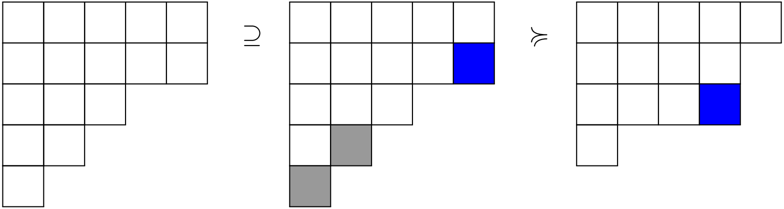

$\lambda = (5,5,3,2,1) \quad \supseteq \quad \mu = (5,5,3,1) \quad \succcurlyeq \quad \nu = (5,4,4,1)$

Containment = removing boxes
Dominance = lowering boxes
Weak dominance = both

Lemma

If $\lambda \succcurlyeq_{\mathrm{w}} \nu$, then $\exists \mu$ such that

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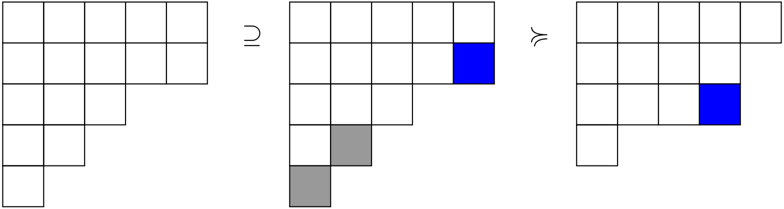

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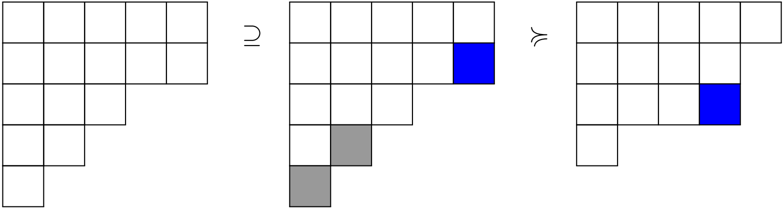

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If $\lambda \succcurlyeq_{\mathrm{w}} \nu$, then $\exists \mu$ such that

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$$\lambda \succcurlyeq \mu \iff b_\lambda(x) \geqslant b_\mu(x), \quad x \in [0,\infty)^n \tag{*}$$

$$\lambda \succcurlyeq_{\mathrm{w}} \mu \iff b_\lambda(x+\mathbf{1}) \geqslant b_\mu(x+\mathbf{1}), \quad x \in [0,\infty)^n \tag{**}$$

$$\lambda \supseteq \mu \iff b_\lambda(x+\mathbf{1}) - b_\mu(x+\mathbf{1}) \text{ is } b\text{-positive} \tag{***}$$

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Theorem

Let $b_\lambda(x)$ be evaluation positive, namely, $b_\lambda(x) \geqslant 0$, $x \in [0,\infty)^n$. Then for the $\Longrightarrow$ direction: $(*) + (***) \Longrightarrow (**)$.

$$\lambda \succcurlyeq \mu \iff b_\lambda(x) \geqslant b_\mu(x), \quad x \in [0,\infty)^n \tag{*}$$

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Proof.

Assume $\lambda \succcurlyeq_{\mathrm{w}} \nu$. By the lemma, $\exists \mu$ such that $\lambda \supseteq \mu \succcurlyeq \nu$.

$$b_\lambda(x+\mathbf{1}) - b_\nu(x+\mathbf{1}) = \big(b_\lambda(x+\mathbf{1}) - b_\mu(x+\mathbf{1})\big) + \big(b_\mu(x+\mathbf{1}) - b_\nu(x+\mathbf{1})\big)$$

By (***), the former $b_\lambda(x+\mathbf{1}) - b_\mu(x+\mathbf{1})$ expands positively into $b_\xi(x)$, which is evaluation positive.
By (*), the latter $b_\mu(x+\mathbf{1}) - b_\nu(x+\mathbf{1})$ is evaluation positive. $\square$

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Theorem

Let $b_\lambda(x)$ be evaluation positive, namely, $b_\lambda(x) \geqslant 0$, $x \in [0,\infty)^n$. Then for the $\Longrightarrow$ direction: (*) + (***) $\Longrightarrow$ (**).

In particular, (**) holds for $b_\lambda = \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})}, \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}, \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})}$, in addition to Khare–Tao's result for $\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}$.

$$\lambda \succcurlyeq \mu \iff b_\lambda(x) \geqslant b_\mu(x), \quad x \in [0,\infty)^n \qquad (*)$$

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(*) and (**) hold for $b_\lambda = \frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})}, \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}, \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}, \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})}$

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Conjecture (C.–Sahi 2024)

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Theorem (C.–Sahi 2024)

For $b_\lambda = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}$, the $\Longleftarrow$ direction in (*) and in (**) is true.
Also, the $\Longrightarrow$ direction in (*) implies the $\Longrightarrow$ direction in (**).

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Theorem (C.–Khare–Sahi 2025)

If λ and μ have at most two parts, then (*) holds for $b_\lambda = \frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)}$.

Sketch of Proof.

All such partitions are in the form $\lambda = (a+b, a)$. For $n = 2$, we have

$$P_{(a+b,a)}(x_1, x_2; \alpha) = (x_1 x_2)^a \cdot P_{(b,0)}(x_1, x_2; \alpha).$$

It suffices to prove for $\lambda = (d, 0) \succcurlyeq \mu = (d-1, 1)$, which is done by explicit computation. For $n > 2$, we use the following lemma. $\square$

Lemma (C.–Khare–Sahi 2025)

For Jack polynomials, if (*) holds for n variables, then it holds for $n+1$ variables.

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For Jack polynomials, if (*) holds for n variables, then it holds for $n+1$ variables.

Let $M_\lambda(x) = \frac{m_\lambda(x)}{m_\lambda(\mathbf{1})}$.
The **Muirhead cone** $\mathcal{M}_C$ consists of non-negative combinations of

$$\{M_\lambda - M_\mu \,|\, \lambda \supseteq \mu\}$$

and **Muirhead semiring** $\mathcal{M}_S$ consists of non-negative combinations of products of

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By Muirhead's inequality, functions in $\mathcal{M}_C$ and $\mathcal{M}_S$ are evaluation positive.
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Cuttler–Greene–Skandera showed that for $\lambda \succcurlyeq \mu$,

$$\frac{e_{\lambda'}(x)}{e_\lambda(\mathbf{1})} - \frac{e_{\mu'}(x)}{e_\mu(\mathbf{1})}, \frac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x)}{p_\mu(\mathbf{1})} \in \mathcal{M}_S \tag{13}$$

Conjecture (C.–Khare–Sahi 2025)

If $\lambda \succcurlyeq \mu$, then $\frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)} - \frac{P_\mu(x;\alpha)}{P_\mu(\mathbf{1};\alpha)} \in \mathcal{M}_S$.

Theorem (C.–Khare–Sahi 2025)

For $n = 2$, we have a stronger result: If $\lambda \succcurlyeq \mu$, then $\frac{P_\lambda(x;\alpha)}{P_\lambda(\mathbf{1};\alpha)} - \frac{P_\mu(x;\alpha)}{P_\mu(\mathbf{1};\alpha)} \in \mathcal{M}_C$.

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(*) *holds for* $b_\lambda = \frac{P_\lambda(x;q,t)}{P_\lambda(\mathbf{1};q,t)}$.

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Thank you!

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