Differential operators for Macdonald's hypergeometric functions

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Slides: https://sites.math.rutgers.edu/~hc813/

Euler (1769) and Gauss (1812) were the first to study the following differential equation and its series solution:

$$z(1-z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+(c-(a+b+1)z)\frac{\mathrm{d}F}{\mathrm{d}z}-abF=0. \tag{1}$$

DISQUISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$

PARS PRIOR

AUCTORE

CAROLO FRIDERICO GAUSS

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Now known as the *Gauss hypergeometric function/series* is

$$\begin{aligned} F(a,b;c;z) &= 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} \\ &\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}, \end{aligned} \tag{2}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol*.

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where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol*. Then $F(a,b;c;z)$ is the unique solution of Eq. (1) subject to the condition that $F(z)$ is analytic at $z=0$ and $F(0)=1$.

One natural way to generalize F is to allow more parameters. Let $\underline{a} = (a_1, \ldots, a_p)$ and $\underline{b} = (b_1, \ldots, b_q)$, then

$$ {}_pF_q(\underline{a}; \underline{b}; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

Q: Is there a differential equation that characterizes ${}_pF_q$?
A: Yes, see [A. Erdélyi, Higher Transcendental Functions (the Bateman Manuscript Project)]

$$ \left(z\frac{\mathrm{d}}{\mathrm{d}z} \prod_{k=1}^{q} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1 \right) - z \prod_{k=1}^{p} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_k \right) \right) (F) = 0, $$

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Since the '50s, Bochner, Herz, Constantine, James and Muirhead, developed the theory of hypergeometric function with matrix argument, namely, defined on real symmetric positive-definite $n \times n$ matrices.

Such generalizations are of great importance in multivariate statistics, random matrix, and even number theory.

Constantine showed that such hypergeometric functions can be written as a series of *Zonal polynomials*, which are Zonal spherical function of the Gelfand pair $(\mathrm{GL}_n(\mathbb{R}), \mathrm{O}_n)$

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Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
A *partition* (of length at most n) is $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$.
The *Schur polynomial* s_λ is defined as

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\det\left(x_i^{n-j}\right)}. \tag{3}$$

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Macdonald's hypergeometric functions, in the Schur case, are

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where $(a)_\lambda$ is a generalization of $(a)_m$ and h_λ is the product of hook-lengths.

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$$ h_{4421}=7\cdot5\cdot3\cdot2\cdot6\cdot4\cdot2\cdot1\cdot3\cdot1\cdot1=30240 $$

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Known Results

- **Q**: Differential operator that characterizes Constantine's ${}_pF_q(\underline{a}; \underline{b}; x; \alpha = 2)$ and ${}_pF_q(\underline{a}; \underline{b}; x, y; \alpha = 2)$ (Zonal case)?
  A: [Muirhead, '70] and [Constantine–Muirhead, '72] solved ${}_2F_1$ and degenerate cases.
  [Fujikoshi, '75] solved ${}_3F_2$ and ${}_2F_2$.
- **Q**: What about Macdonald's ${}_pF_q(\underline{a}; \underline{b}; x; \alpha)$ and ${}_pF_q(\underline{a}; \underline{b}; x, y; \alpha)$ (Jack case)?
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  [Yan, '92] and [Kaneko, '93] solved ${}_2F_1(a, b; c; x; \alpha)$.

  To the best of our knowledge, there are no further results.

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Our work

We find, for arbitrary p and q, the following differential operators

- a lowering operator $\mathcal{L}_q$,
- a raising operator ${}_p\mathcal{R}$,
- two eigen-operators ${}_p\mathcal{M}$ and $\mathcal{N}_q$,

such that ${}_pF_q(\underline{a}; \underline{b}; x, y; \alpha)$ is the unique solution of

$$(\mathcal{L}_q^{(x)} - {}_p\mathcal{R}^{(y)})(F(x, y)) = 0, \quad (6)$$

and ${}_pF_q(\underline{a}; \underline{b}; x; \alpha)$ is the unique solution of

$$(\mathcal{L}_q - {}_p\mathcal{M})(F(x)) = 0, \quad (7)$$

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For comparison, in the classical case, ${}_pF_q(\underline{a};\underline{b};z)$

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{k=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_k-1\right) - z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_k\right)\right)(F) = 0.$$

Our work

The lowering operator $\mathcal{L}_q$ is constructed using the divergence operator $E_1 = \sum_i \partial_i$ and the Laplace–Beltrami operator

$$\Box = \sum_i \frac{1}{2} x_i^2 \partial_i + \frac{1}{\alpha} \sum_{i \neq j} \frac{x_i x_j}{x_i - x_j} \partial_i.$$

The raising operator ${}_p\mathcal{R}$ is constructed using multiplication by $e_1 = \sum_i x_i$ and the Laplace–Beltrami operator $\Box$.

Two eigen-operators ${}_p\mathcal{M}$ and $\mathcal{N}_q$ are constructed using the Debiard–Sekiguchi operators, which are commuting differential operators that act diagonally on Jack polynomials.

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Two eigen-operators ${}_p\mathcal{M}$ and $\mathcal{N}_q$ are constructed using the Debiard–Sekiguchi operators, which are commuting differential operators that act diagonally on Jack polynomials.

Future

Macdonald also introduced a Macdonald polynomial analogue of the hypergeometric functions:

$$
\begin{aligned}
{}_r\Phi_s(\underline{a};\underline{b};x;q,t) &= \sum_{\lambda} \frac{(a_1)_\lambda \cdots (a_r)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} t^{n(\lambda)} \frac{J_\lambda(x;q,t)}{j_\lambda}, \\
{}_r\Phi_s(\underline{a};\underline{b};x,y) &= \sum_{\lambda} \frac{(a_1)_\lambda \cdots (a_r)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} t^{n(\lambda)} \frac{J_\lambda(x;q,t) J_\lambda(y;q,t)}{j_\lambda J_\lambda(1,t,\ldots,t^{n-1};q,t)},
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where $(a)_\lambda = (a;q,t)_\lambda$ is the (q,t)-Pochhammer symbol.

Q: Find q-difference operator that characterizes ${}_r\Phi_s$.

A: Work in progress with Siddhartha Sahi.

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Thank you!

I will be on job market this Fall.
email: hc813@math.rutgers.edu
slides: https://sites.math.rutgers.edu/~hc813/

In the 1980s, Macdonald introduced a Jack polynomial analogue of hypergeometric functions:

$$ {}_pF_q(\underline{a};\underline{b};x;\alpha) = \sum_\lambda \frac{(a_1;\alpha)_\lambda \cdots (a_p;\alpha)_\lambda}{(b_1;\alpha)_\lambda \cdots (b_q;\alpha)_\lambda} \alpha^{|\lambda|} \frac{J_\lambda(x;\alpha)}{j_\lambda}, \tag{9} $$

$$ {}_pF_q(\underline{a};\underline{b};x,y;\alpha) = \sum_\lambda \frac{(a_1;\alpha)_\lambda \cdots (a_p;\alpha)_\lambda}{(b_1;\alpha)_\lambda \cdots (b_q;\alpha)_\lambda} \alpha^{|\lambda|} \frac{J_\lambda(x;\alpha)J_\lambda(y;\alpha)}{j_\lambda J_\lambda(\mathbf{1}_n)}, \tag{10} $$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
The Zonal case (Jack with $\alpha = 2$) was first introduced in 1960s by Constantine.

The Pochhammer symbol $(a)_m = a(a+1)\cdots(a+m-1)$ can be represented by the tableau

$$\boxed{0}\boxed{1}\ \cdots\ \boxed{m-1}$$

For a partition, say, $\lambda = 4421$, we use the content:

0	1	2	3
−1	0	1	2
−2	−1		
−3			

$(a)_\lambda = a^2(a+1)^2(a+2)^2(a+3)(a-1)^2(a-2)(a-3)$.
The *Pochhammer symbol* is defined as

$$(a)_\lambda = \prod_{(i,j)\in\lambda} (a + j - i)\,,$$

Jack case: use α-content $j - 1 - (i-1)/\alpha$.