Binomial Coefficients and Littlewood–Richardson Coefficients for Interpolation Polynomials

Hong Chen

Rutgers

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Randonness and Lie-Theoretic Structures
University of Virginia

Interpolation Polynomials

Let $\mathcal{P}_n$ be the set of all **partitions** of length at most n, i.e., $\mathcal{P}_n = \{ \lambda \in \mathbb{N}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0 \}$. Write $|\lambda| \coloneqq \sum \lambda_i$.

There are four families of interpolation Jack and Macdonald polynomials, developed by Knop–Sahi (type A) and Okounkov (type BC). Write AJ for the interpolation Jack polynomials of type A, and similarly for AM, BJ, BM.

Definition (Interpolation Polynomials)

The unital **interpolation polynomial** is the unique $\mathcal{W}$-symmetric function that satisfies the following vanishing and normalization condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad \forall \lambda \in \mathcal{P}_n,\ |\lambda| \leqslant |\mu|, \tag{1}$$

$$\deg h_\mu \leqslant \begin{cases} |\mu|, & \mathcal{F} = A\mathrm{J}, A\mathrm{M}, B\mathrm{M}; \\ 2|\mu|, & \mathcal{F} = B\mathrm{J}. \end{cases} \tag{2}$$

Combinatorial Formulas

$$P_\lambda^{\mathrm{monic},\mathrm{J}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},A\mathrm{J}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau\right)\right),$$

$$h_\lambda^{\mathrm{monic},B\mathrm{J}}(x;\tau,\alpha) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)}^2 - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau + \alpha\right)^2\right),$$

$$P_\lambda^{\mathrm{monic},\mathrm{M}}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},A\mathrm{M}}(x;q,t) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)}\right),$$

$$h_\lambda^{\mathrm{monic},B\mathrm{M}}(x;q,t,a) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \Bigg(x_{T(s)} + x_{T(s)}^{-1} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a - \left(q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a\right)^{-1}\Bigg)$$

$\overline{\lambda} = \lambda + \tau\delta,\ \lambda + \tau\delta + \alpha,\ q^\lambda t^\delta,\ aq^\lambda t^\delta$ for AJ, BJ, AM, BM, where $\delta = (n-1, n-2, \ldots, 1, 0)$.

Extra Vanishing Property and Binomial Coefficients

It is a surprising fact that the interpolation polynomials vanish at more points than required in the definition.

Proposition (Extra Vanishing Property)

$$h_\mu(\overline{\lambda}) = 0, \quad \textit{unless} \quad \lambda \supseteq \mu.$$

Write $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$, $1 \leqslant i \leqslant n$.

Definition ((Adjacent) Binomial Coefficients)

$$b_{\lambda\mu} = \binom{\lambda}{\mu} = h_\mu(\overline{\lambda}), \quad a_{\lambda\mu} := \begin{cases} b_{\lambda\mu}, & \lambda :\supset \mu; \\ 0, & \text{otherwise}, \end{cases}$$

Write $\lambda :\supset \mu$ if $\lambda \supseteq \mu$ and $|\lambda| = |\mu| + 1$.
Binomial coefficients appear in Okounkov–Olshanski's binomial formula.

Weighted Sum Formula

Theorem (Sahi 2011, Weighted Sum Formula for $b_{\lambda\mu}$)

Let $\lambda \supseteq \mu$, and $k \coloneqq |\lambda| - |\mu|$. Then in the cases of AJ, AM*,*

$$b_{\lambda\mu} = \sum_{\zeta \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}(\zeta) \prod_{i=0}^{k-1} a_{\zeta_i \zeta_{i+1}}, \tag{3}$$

$$\mathrm{wt}(\zeta) \coloneqq \prod_{i=0}^{k-1} \frac{\left\|\overline{\zeta_i}\right\| - \left\|\overline{\zeta_{i+1}}\right\|}{\left\|\overline{\zeta_0}\right\| - \left\|\overline{\zeta_{i+1}}\right\|}. \tag{4}$$

where the sum is over all the chains $\zeta = (\zeta_0, \ldots, \zeta_k)$ with

$$\lambda = \zeta_0 :\supset \zeta_1 :\supset \cdots :\supset \zeta_{k-1} :\supset \zeta_k = \mu,$$

and $\left\|\overline{\lambda}\right\|$ can be taken to be $b_{\lambda\varepsilon_1} = h_{\varepsilon_1}(\overline{\lambda})$, with $\varepsilon_1 = (1, 0, \ldots, 0)$.

Main Results: Binomial Coefficients

Theorem (C–Sahi 2024, Theorem A)

The weighted sum formula Eq. (3) holds for BJ, BM as well.

For each family, we define a **cone of positivity**, $\mathbb{F}^+ \subseteq \mathbb{F}$.
For AJ, $\mathbb{F} = \mathbb{Q}(\tau)$ and $\mathbb{F}^+ := \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{N}[\tau] \setminus 0 \right\}$.
For AM, $\mathbb{F} = \mathbb{Q}(q, t)$ and $\mathbb{F}^+$ consists of functions $f(q, t) > 0$ when $0 < q, t < 1$.

Theorem (C–Sahi 2024, Theorem B, Positivity)

The binomial coefficients $b_{\lambda\mu} \in \mathbb{F}^+$ if and only if $\lambda \supseteq \mu$.

Theorem (C–Sahi 2024, Theorem C, Monotonicity)

The binomial coefficients $b_{\lambda\nu} - b_{\mu\nu} \in \mathbb{F}^+$ if $\lambda \supsetneq \mu \supseteq \nu \neq \mathbf{0}$.

Applications

Theorem (Okounkov–Olshanski's Binomial Formula)

Let P_λ be the monic Jack polynomial, and $b_{\lambda\mu}$ be the binomial coefficients for the family AJ. Then

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} = \sum_{\mu\subseteq\lambda} b_{\lambda\mu}\frac{P_\mu(x)}{P_\mu(\mathbf{1})}, \tag{5}$$

where $\mathbf{1} = (1^n) = (1,\ldots,1)$.

Theorem (C–Sahi 2024, Theorem F)

TFAE:

1. λ *contains* μ;
2. $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is* ***Schur positive;***
3. $\dfrac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \dfrac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})}$ *is* ***Jack positive.***

Related Results

Recall that for n-tuples λ, μ, we say λ **weakly majorizes (weakly dominates)** μ if $\sum_{i=1}^r \lambda_i \geqslant \sum_{i=1}^r \mu_i$, for all $1 \leqslant r \leqslant n$; λ **majorizes** μ if, in addition, $|\lambda| = |\mu|$.

Theorem (Cuttler–Greene–Skandera 2011, Sra 2016)

Let $|\lambda| = |\mu|$. Then λ majorizes μ if and only if

$$\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} \geqslant \frac{s_\mu(x)}{s_\mu(\mathbf{1})}, \quad \forall x \in [0, \infty)^n. \tag{6}$$

Theorem (Khare–Tao 2018)

λ weakly majorizes μ if and only if

$$\frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n. \tag{7}$$

Generalization

Conjecture

Let P_λ *be the Jack polynomial and* $\mathbb{F}^+$ *be given as above for AJ.*

- *(CGS Conjecture for Jack polynomials) Suppose* $|\lambda| = |\mu|$*.* λ *majorizes* μ *if and only if*

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \in \mathbb{F}^+ \cup 0, \quad \forall x \in [0, \infty)^n. \tag{8}$$

- *(KT Conjecture for Jack polynomials)* λ *weakly majorizes* μ *if and only if*

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})} \in \mathbb{F}^+ \cup 0, \quad \forall x \in [0, \infty)^n. \tag{9}$$

Littlewood–Richardson Coefficients

Definition (Littlewood–Richardson (LR) Coefficients)

The **Littlewood–Richardson (LR) coefficients** are defined by the product expansion

$$h_\mu(x)h_\nu(x) = \sum_\lambda c_{\mu\nu}^\lambda h_u(x).$$

Because the top degree terms of the interpolation Jack/Macdonald polynomials are related to be ordinary Jack/Macdonald polynomials, $c_{\mu\nu}^\lambda$ generalizes the corresponding coefficients for Jack/Macdonald polynomials.

Main Results: LR Coefficients

Theorem (C–Sahi 2024, Theorem D)

$$c_{\mu\nu}^{\lambda} = \sum_{\boldsymbol{\zeta} \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}_{\nu}^{\mathrm{LR}}(\boldsymbol{\zeta}) \prod_{i=0}^{k-1} a_{\boldsymbol{\zeta}_i \boldsymbol{\zeta}_{i+1}}, \tag{10}$$

$$\mathrm{wt}_{\nu}^{\mathrm{LR}}(\boldsymbol{\zeta}) := \sum_{j=0}^{k} \frac{\prod_{0 \leqslant i \leqslant k-1} \left(\left\| \overline{\boldsymbol{\zeta}_i} \right\| - \left\| \overline{\boldsymbol{\zeta}_{i+1}} \right\| \right)}{\prod_{\substack{0 \leqslant i \leqslant k \\ i \neq j}} \left(\left\| \overline{\boldsymbol{\zeta}_j} \right\| - \left\| \overline{\boldsymbol{\zeta}_i} \right\| \right)} b_{\boldsymbol{\zeta}_j \nu}. \tag{11}$$

Theorem (C–Sahi 2024, Theorem E)

Assume $\lambda :\supset \mu$, then the ***adjacent LR coefficient*** $c_{\mu\nu}^{\lambda}$ *lies in the cone of positive $\mathbb{F}^{+}$ if $\lambda \supseteq \nu \neq \mathbf{0}$ and is 0 otherwise.*

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Thanks for listening.