Symmetric Polynomials and Interpolation Polynomials

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What are *interpolation polynomials*?

They are symmetric polynomials and generalizations of the ordinary Schur/Jack/Macdonald polynomials: they are *inhomogeneous*, whose top degree terms correspond to the ordinary polynomials.
They *interpolate* the Kronecker delta function (up to certain rank).
They are developed by Knop–Sahi (type A) and Okounkov (type BC), also called *shifted* polynomials by Okounkov.

Today I will introduce these polynomials and the so-called *generalized binomial coefficients* and *Littlewood–Richardson coefficients* associated to them. I will also talk about some of our recent results, including two weighted sum formulas, positivity and monotonicity, and an application to Jack positivity.

The slides of my talk can be found on
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Partitions

A **partition** λ is a finite or infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ of non-negative integers in weakly decreasing order:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots$$

with only finitely many non-zero terms.
The non-zero λ_i are called **parts**; the number of parts is called the **length** of λ, denoted by $\ell(\lambda)$; the sum of the parts is called the **weight** or **size** of λ, denoted by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$.
Some reasons that partitions are important:

- Conjugacy classes of the symmetric group S_n are indexed by partitions;
- Many(any) natural bases of the ring of symmetric polynomials are indexed by partitions.

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Symmetric Polynomials

Fix $n \geqslant 1$, the number of variables, and let $\mathcal{P}_n$ be the set of partitions of length at most n. Let $\Lambda = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ and $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$. In the following, $\mathbb{F}$ will be $\mathbb{Q}$, $\mathbb{Q}(\alpha)$, or $\mathbb{Q}(q,t)$, where α, q, t are indeterminates.

The monomial basis: $m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$, naturally a $\mathbb{Z}$-basis of Λ.

The power sum basis: $p_r = \sum_i x_i^r$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$. $\{p_\lambda\}$ is not a $\mathbb{Z}$-basis of Λ, for example, $m_{11} + m_2 = \frac{1}{2}(p_{11} + p_2)$, but a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}}$.

The Schur basis s_λ. There are many ways to define them:

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})} = \det(h_{\lambda_i - i + j}) = \sum_T \prod_{s \in \lambda} x_{T(s)}. \tag{1}$$

The last one yields the so-called **Kostka number** $K_{\lambda\mu}$,

$$s_\lambda = \sum_{\mu \leqslant \lambda} K_{\lambda\mu} m_\mu. \tag{2}$$

which can be combinatorially interpreted as the number of semi-standard tableaux of shape λ and weight μ.

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Orthogonal Polynomials

Schur polynomials can even be defined abstractly. Define an inner product $\langle \cdot, \cdot \rangle$ on Λ by the following:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda, \tag{3}$$

where z_λ is some number. Then s_λ is the unique polynomial satisfying:

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu, \quad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}. \tag{4}$$

More generally, one can define two other inner products:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}, \quad \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{5}$$

Then there is a unique $P_\lambda^{(\alpha)}(x)$ in $\Lambda_{\mathbb{Q}(\alpha)}$ and a unique $P_\lambda(x;q,t)$ in $\Lambda_{\mathbb{Q}(q,t)}$, called **Jack** and **Macdonald** polynomials, satisfying

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu, \quad \langle P_\lambda, P_\mu \rangle = 0, \ \lambda \neq \mu. \tag{6}$$

When $\alpha = 1$ or $q = t$, P_λ becomes s_λ.

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Interpolation Polynomials

There are four families of interpolation Jack and Macdonald polynomials, developed by Knop–Sahi (type A) and Okounkov (type BC). Denote by AJ, AM the type A interpolation polynomials and similarly BJ, BM for type BC. They can uniformly defined as follows:

Definition (Interpolation Polynomials)

The unital **interpolation polynomial** is the unique $\mathcal{W}$-symmetric function that satisfies the following interpolation condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad \forall \lambda \in \mathcal{P}_n, \ |\lambda| \leqslant |\mu|, \tag{7}$$

$$\deg h_\mu = \begin{cases} |\mu|, & A\text{J}, A\text{M}, B\text{M}; \\ 2|\mu|, & B\text{J}, \end{cases} \tag{8}$$

where, $\overline{\lambda}$ is some shifting of λ, depending on the family.

Combinatorial Formulas

Okounkov showed that they admit the following combinatorial formulas: (The Jack parameter τ corresponds to Macdonald's $1/\alpha$)

$$P_\lambda^{\mathrm{monic,J}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},AJ}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau \right) \right),$$

$$h_\lambda^{\mathrm{monic},BJ}(x;\tau,\alpha) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x^2_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau + \alpha \right)^2 \right),$$

$$P_\lambda^{\mathrm{monic,M}}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},AM}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \left(x_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} \right),$$

$$h_\lambda^{\mathrm{monic},BM}(x;q,t,a) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \Bigg(x_{T(s)} + x^{-1}_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a - \left(q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a \right)^{-1} \Bigg)$$

where all the sums are over column-strict reverse tableaux $T : \lambda \to [n]$, i.e., weakly decreasing along the rows and strictly decreasing along the columns.

$\overline{\lambda} = \lambda + \tau\delta,\ \lambda + \tau\delta + \alpha,\ q^\lambda t^\delta,\ a q^\lambda t^\delta$ for AJ, BJ, AM, BM, where $\delta = (n-1, \ldots, 1, 0)$.

Examples

Let $n = 2$, $\mu = (3,2)$, one can find $s_\mu = P^{\mathrm{J}}_\mu = P^{\mathrm{M}}_\mu = m_\mu = x_1^3 x_2^2 + x_1^2 x_2^3$,

$$\begin{aligned} h_\mu^{\mathrm{monic},AJ} &= x_2 x_1 (x_2 - 1)(x_1 - 1)(x_1 - 2 - \tau) \\ &\quad + x_2 x_1 (x_2 - 1)(x_1 - 1)(x_2 - 2) \\ &= x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 + x_2 - \tau - 4) \end{aligned}$$

$$\begin{aligned} h_\mu^{\mathrm{monic},AM} &= (x_2 - 1)(x_1 - 1)(x_2 - q)(x_1 - q)(x_1 - q^2 t) \\ &\quad + (x_2 - 1)(x_1 - 1)(x_2 - q)(x_1 - q)(x_2 - q^2) \\ &= (x_1 - 1)(x_2 - 1)(x_1 - q)(x_2 - q)(x_1 + x_2 - q^2 t - q^2) \end{aligned}$$

The defining condition involves the following 12 partitions:

$(0,0), (1,0), (1,1), (2,0), (2,1), (3,0), (2,2), (3,1), (4,0), (3,2), (4,1), (5,0)$.

Note that $\overline{(\lambda_1, \lambda_2)} = (\lambda_1 + \tau, \lambda_2), (q^{\lambda_1} t, q^{\lambda_2})$. Because of the factorizations, one can easily see that h_μ vanishes at all but $\overline{(3,2)}$. Moreover, h_μ also vanishes at $\overline{(m,0)}$ and $\overline{(m-1,1)}$ $\forall m \geqslant 6$.
Warning: In most cases, no complete factorization!

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Extra Vanishing Property and Binomial Coefficients

In general, it is a surprising fact that the interpolation polynomials vanish at more points than required in the definition.

Proposition (Knop–Sahi, Okounkov '90s, Extra Vanishing Property)

$$h_\mu(\overline{\lambda}) = 0, \quad \textit{unless} \quad \lambda \supseteq \mu.$$

Write $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$, $1 \leqslant i \leqslant n$.

Definition ((Adjacent) Binomial Coefficients)

$$b_{\lambda\mu} = \binom{\lambda}{\mu} = h_\mu(\overline{\lambda}), \quad a_{\lambda\mu} := \begin{cases} b_{\lambda\mu}, & \lambda :\supset \mu; \\ 0, & \text{otherwise}, \end{cases}$$

Write $\lambda :\supset \mu$ if $\lambda \supseteq \mu$ and $|\lambda| = |\mu| + 1$.
Binomial coefficients appear in Okounkov–Olshanski's binomial formula.
Some combinatorial formulas for adjacent binomial coefficients are given in [C-Sahi, Prop 4.3].

Extra Vanishing Property and Binomial Coefficients

In general, it is a surprising fact that the interpolation polynomials vanish at more points than required in the definition.

Proposition (Knop–Sahi, Okounkov '90s, Extra Vanishing Property)

$$h_\mu(\overline{\lambda}) = 0, \quad \textit{unless} \quad \lambda \supseteq \mu.$$

Write $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$, $1 \leqslant i \leqslant n$.

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Temporarily, consider the family AJ and AM only.

Lemma (Sahi '11, Pieri Rule)

Let $\varepsilon_1 = (1, 0^{n-1}) = (1, 0, \ldots, 0)$ *and* $\mu \in \mathcal{P}_n$. *Then*

$$\big(h_{\varepsilon_1}(x) - h_{\varepsilon_1}(\overline{\mu})\big) \cdot h_\mu(x) = \sum_{\nu :\supset \mu} \big(h_{\varepsilon_1}(\overline{\nu}) - h_{\varepsilon_1}(\overline{\mu})\big)\, a_{\nu\mu} h_\nu(x). \tag{9}$$

Proof.

Compare the two sides at $\overline{\lambda}$ for $|\lambda| \leqslant |\mu| + 1$. □

Let $\left\|\overline{\lambda}\right\| = b_{\lambda\varepsilon_1} = h_{\varepsilon_1}(\overline{\lambda})$. Write $A = \left(a_{\lambda\mu}\right), B = \left(b_{\lambda\mu}\right), Z = \left(\|\overline{\mu}\|\, \delta_{\lambda\mu}\right)$.

Theorem (Sahi '11, Recursion Formula)

1. *The following recursions characterize $b_{\lambda\mu}$:*

$$(i)\ \ b_{\lambda\lambda} = 1; (ii)\ \left(\left\|\overline{\lambda}\right\| - \|\overline{\mu}\|\right) b_{\lambda\mu} = \sum_{\nu :\supset \mu} b_{\lambda\nu}\left(\|\overline{\nu}\| - \|\overline{\mu}\|\right) a_{\nu\mu}. \quad (10)$$

2. *The matrices A, B, Z satisfy the commutation relations:*

$$(i)\ \ [Z, B] = B[Z, A], \quad (ii)\ \ [Z, B^{-1}] = -[Z, A]B^{-1}. \quad (11)$$

Proof.

Evaluate the Pieri rule at $\overline{\lambda}$. □

Theorem (Sahi '11, Weighted Sum Formula for $b_{\lambda\mu}$)

Let $\lambda \supseteq \mu$, and $k := |\lambda| - |\mu|$.

$$b_{\lambda\mu} = \sum_{\boldsymbol{\zeta} \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}(\boldsymbol{\zeta}) \prod_{i=0}^{k-1} a_{\zeta_i \zeta_{i+1}}, \tag{12}$$

$$\mathrm{wt}(\boldsymbol{\zeta}) := \prod_{i=0}^{k-1} \frac{\left\| \overline{\zeta_i} \right\| - \left\| \overline{\zeta_{i+1}} \right\|}{\left\| \overline{\zeta_0} \right\| - \left\| \overline{\zeta_{i+1}} \right\|}. \tag{13}$$

where the sum is over all the chains $\boldsymbol{\zeta} = (\zeta_0, \ldots, \zeta_k)$ with

$$\lambda = \zeta_0 :\supset \zeta_1 :\supset \cdots :\supset \zeta_{k-1} :\supset \zeta_k = \mu,$$

(i.e., standard tableaux of skew shape λ/μ).

Main Results: Binomial Coefficients

Theorem (C–Sahi, Lemma 3.1, Theorem 3.2, Theorem A)

The aforementioned Pieri rule, recursion formula, and weighted sum formula hold for BJ, BM *as well.*

For each family, we define a **cone of positivity** $\mathbb{F}^+$ in the base field $\mathbb{F}$. For example:

For AJ, $\mathbb{F} = \mathbb{Q}(\tau)$ and $\mathbb{F}^+ := \left\{ \frac{f}{g} \;\middle|\; f, g \in \mathbb{N}[\tau] \setminus 0 \right\}$. In particular, for $f \in \mathbb{F}^+$, $f(\tau) > 0$, when $\tau > 0$.

For AM, $\mathbb{F} = \mathbb{Q}(q, t)$ and $\mathbb{F}^+$ consists of functions $f(q, t) > 0$ when $0 < q, t < 1$.

Theorem (C–Sahi, Theorem B, Positivity)

The binomial coefficients $b_{\lambda\mu} \in \mathbb{F}^+$ *if and only if* $\lambda \supseteq \mu$.

Theorem (C–Sahi, Theorem C, Monotonicity)

The binomial coefficients $b_{\lambda\nu} - b_{\mu\nu} \in \mathbb{F}^+$ *if* $\lambda \supsetneq \mu \supseteq \nu \neq \mathbf{0}$.

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An Application

Theorem (Okounkov–Olshanski '97, Binomial Formula)

Let P_λ be the Jack polynomial, and $b_{\lambda\mu}$ be the binomial coefficients for the family AJ. Write $\mathbf{1} = (1^n) = (1, \ldots, 1)$. Then

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} = \sum_{\nu \subseteq \lambda} b_{\lambda\nu} \frac{P_\nu(x)}{P_\nu(\mathbf{1})}. \tag{14}$$

Theorem (C–Sahi, Theorem F)

TFAE:

1. λ *contains* μ*;*
2. $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is Schur positive;*
3. $\dfrac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \dfrac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})}$ *is Jack positive.*

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Related Results

Recall that we say λ **weakly majorizes** (**weakly dominates**) μ if $\sum_{i=1}^{r} \lambda_i \geqslant \sum_{i=1}^{r} \mu_i$, for all $1 \leqslant r \leqslant n$; λ **majorizes** (**dominates**) μ if, in addition, $|\lambda| = |\mu|$.

Theorem (Cuttler–Greene–Skandera '11, Sra '16)

Let $|\lambda| = |\mu|$. Then λ majorizes μ if and only if

$$\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n. \tag{15}$$

Theorem (Khare–Tao '18)

λ weakly majorizes μ if and only if

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Generalization

Conjecture

Let P_λ be the Jack polynomial and $\mathbb{F}_{\mathbb{R}}^{+} = \{ f/g \mid f, g \in \mathbb{R}_{\geqslant 0}[\tau] \setminus 0 \}$.

- *(CGS Conjecture for Jack polynomials) Suppose* $|\lambda| = |\mu|$. λ *majorizes* μ *if and only if*

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \in \mathbb{F}_{\mathbb{R}}^{+} \cup 0, \quad \forall x \in [0, \infty)^n. \tag{17}$$

- *(KT Conjecture for Jack polynomials)* λ *weakly majorizes* μ *if and only if*

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})} \in \mathbb{F}_{\mathbb{R}}^{+} \cup 0, \quad \forall x \in [0, \infty)^n. \tag{18}$$

Littlewood–Richardson Coefficients

Definition (Littlewood–Richardson Coefficients)

The **Littlewood–Richardson coefficients** are defined by the product expansion

$$h_\mu(x)h_\nu(x) = \sum_\lambda c_{\mu\nu}^\lambda h_\lambda(x).$$

Because the top degree terms of the interpolation polynomials are related to be ordinary ones, these $c_{\mu\nu}^\lambda$ generalize the usual LR coefficients for Jack/Macdonald polynomials.
They are good for induction or recursion purpose [Sahi '11], as interpolation polynomials are inhomogeneous.

Main Results: LR Coefficients

Theorem (C–Sahi, Theorem D, Weighted Sum Formula for $c_{\mu\nu}^{\lambda}$)

$$c_{\mu\nu}^{\lambda} = \sum_{\boldsymbol{\zeta} \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}_{\nu}^{\mathrm{LR}}(\boldsymbol{\zeta}) \prod_{i=0}^{k-1} a_{\zeta_i \zeta_{i+1}}, \tag{19}$$

$$\mathrm{wt}_{\nu}^{\mathrm{LR}}(\boldsymbol{\zeta}) := \sum_{j=0}^{k} \frac{\prod\limits_{0 \leqslant i \leqslant k-1} \left(\left\| \overline{\zeta_i} \right\| - \left\| \overline{\zeta_{i+1}} \right\| \right)}{\prod\limits_{\substack{0 \leqslant i \leqslant k \\ i \neq j}} \left(\left\| \overline{\zeta_j} \right\| - \left\| \overline{\zeta_i} \right\| \right)} b_{\zeta_j \nu}. \tag{20}$$

A special case: when $\lambda = \nu$, $c_{\mu\lambda}^{\lambda} = b_{\lambda\mu}$, and the weighted sum formula for LR coefficients *degenerates* to that for binomial coefficients.
Another special case: when $\lambda :\supset \mu$, $c_{\mu\nu}^{\lambda} = a_{\lambda\mu}(b_{\lambda\nu} - b_{\mu\nu})$.

Theorem (C–Sahi, Theorem E)

Assume $\lambda :\supset \mu$, then the ***adjacent LR coefficient*** *$c_{\mu\nu}^{\lambda}$ lies in the cone of positive $\mathbb{F}^+$ if $\lambda \supseteq \nu \neq \mathbf{0}$ and is 0 otherwise.*

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Thanks for listening.