Binomial formula and interpolation polynomials

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Binomial formula: the classical case

We have the classical Newton's binomial theorem.

Theorem

For non-negative integers m, n, and a variable x,

$$(x+1)^n = \sum_m \binom{n}{m} x^m.$$

In fact, n could be any real number, and x is a real number in a neighborhood of 0.

Question

How to generalize this to symmetric polynomials?

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How to generalize this to symmetric polynomials?

Partitions and symmetric polynomials

Fix $n \geqslant 1$, the number of variables.
A partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

Denote by $\mathcal{P}_n$ the set of such partitions. The length $\ell(\lambda)$ is the number of non-zero parts, and the size is $|\lambda| = \sum \lambda_i$.

Many (any) interesting bases of the ring of symmetric polynomials are indexed by partitions, to name a few, the monomial $m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$, the power-sum $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, and the elementary $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where $p_r = m_{(r)} = \sum_i x_i^r$, $e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$.

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Schur polynomials

The most important basis is perhaps Schur polynomials s_λ.

$$s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})} = \det(e_{\lambda'_i-i+j}) = \sum_T \prod_{s\in\lambda} x_{T(s)}.$$

- For combinatorics, Schur polynomials are generating functions of Young tableaux.
- For rep theory, Schur polynomials correspond bijectively to all irreducible characters of the symmetric groups.

Schur polynomials can also be defined abstractly. Define an inner product $\langle\cdot,\cdot\rangle$ on Λ_n, the ring of symmetric polynomials in n variables, by the following:

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda,$$

where z_λ is some integer. Then s_λ is the unique polynomial satisfying:

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu, \quad \langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}.$$

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Jack and Macdonald polynomials

One can define inner products on $\Lambda_n \otimes \mathbb{Q}(\tau)$ and $\Lambda_n \otimes \mathbb{Q}(q,t)$:

$$\langle p_\lambda, p_\mu \rangle_\tau = \delta_{\lambda\mu} z_\lambda \tau^{-\ell(\lambda)}, \quad \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Then there is a unique $P_\lambda(x;\tau)$ in $\Lambda_n \otimes \mathbb{Q}(\alpha)$ and a unique $P_\lambda(x;q,t)$ in $\Lambda_n \otimes \mathbb{Q}(q,t)$, called **Jack** and **Macdonald** polynomials, satisfying

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu, \quad \langle P_\lambda, P_\mu \rangle = 0, \ \lambda \neq \mu.$$

Jack and Macdonald polynomials are generalizations of Schur and many other polynomials:

$\tau = 1$ or $q = t$: Schur;

$\tau = 0$ or $t = 1$: monomial;

$\tau = \infty$ or $q = 1$: transposed elementary;

$\tau = \frac{1}{2}, 2$: Zonal; $q = 0$: Hall–Littlewood; $t = 0$: q-Whittaker.

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Let $x = (x_1, \dots, x_n)$ and $\mathbf{1} = (1^n) = (1, \dots, 1)$. Okounkov and Olshanski ('97) prove the following for Jack polynomials:

$$(x+1)^n = \sum_m \binom{n}{m} x^m$$

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_\mu \binom{\lambda}{\mu}_\tau \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}$$

$$\binom{n}{m} = \left.\frac{x(x-1)\cdots(x-m+1)}{m!}\right|_{x=n}$$

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These h_μ are called interpolation Jack polynomials [Sahi '94, Knop–Sahi '96], also called shifted Jack polynomials by Okounkov.

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Definition

The *unital interpolation polynomial*, denoted by h_μ, is the unique symmetric function that satisfies the following interpolation condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad \forall \lambda \in \mathcal{P}_n,\ |\lambda| \leqslant |\mu|, \tag{1}$$

$$\deg h_\mu = |\mu|, \tag{2}$$

where $\overline{\lambda}_i = \lambda_i + (n-i)\tau$ in the Jack case, and $\overline{\lambda}_i = q^{\lambda_i} t^{n-i}$ in the Macdonald case.

This normalization is called *unital* as $\binom{\mu}{\mu} = h_\mu(\overline{\mu}) = 1$.
There are other normalizations: *monic*, whose top degree part is the monic Jack/Macdonald P_μ; *integral*, whose top degree part is the integral Jack/Macdonald J_μ.
The above is type A interpolation polynomials, introduced by Knop–Sahi in the '90s. Okounkov also introduced type BC interpolation polynomials, defined by setting $\overline{\lambda}_i = \lambda_i + (n-i)\tau + \alpha$ and $aq^{\lambda_i} t^{n-i}$.

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Okounkov–Olshanski proved the following combinatorial formulas:

$$P_\lambda^{\text{monic,J}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)}, \tag{3}$$

$$h_\lambda^{\text{monic},AJ}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau\right)\right), \tag{4}$$

$$h_\lambda^{\text{monic},BJ}(x;\tau,\alpha) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x^2_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau + \alpha\right)^2\right), \tag{5}$$

$$P_\lambda^{\text{monic,M}}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)}, \tag{6}$$

$$h_\lambda^{\text{monic},AM}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \left(x_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)}\right), \tag{7}$$

$$h_\lambda^{\text{monic},BM}(x;q,t,a) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \Bigg(x_{T(s)} + x^{-1}_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a - \left(q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a\right)^{-1}\Bigg) \tag{8}$$

where all the sums are over column-strict reverse tableaux $T:\lambda\to[n]$, i.e., weakly decreasing along the rows and strictly decreasing along the columns.

Let $n=2$, $\mu=(3,2)$, one can find $s_\mu = P_\mu^{\mathrm{J}} = P_\mu^{\mathrm{M}} = m_\mu = x_1^3x_2^2 + x_1^2x_2^3$,

$$\begin{aligned} h_\mu^{\mathrm{monic},AJ} &= x_2x_1(x_2-1)(x_1-1)(x_1-2-\tau) \\ &\quad + x_2x_1(x_2-1)(x_1-1)(x_2-2) \\ &= x_1x_2(x_1-1)(x_2-1)(x_1+x_2-\tau-4) \end{aligned}$$

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The defining condition involves the following 12 partitions:

$$(0,0),(1,0),(1,1),(2,0),(2,1),(3,0),(2,2),(3,1),(4,0),(3,2),(4,1),(5,0).$$

Note that $\overline{(\lambda_1,\lambda_2)} = (\lambda_1+\tau,\lambda_2),(q^{\lambda_1}t,q^{\lambda_2})$. Because of the factorizations, one can easily see that h_μ vanishes at all but $\overline{(3,2)}$. Moreover, h_μ also vanishes at $\overline{(m,0)}$ and $\overline{(m-1,1)}$ $\forall m\geqslant 6$.

Warning: In most cases, no complete factorization!

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In general, it is a surprising fact that the interpolation polynomials vanish at more points than required in the definition.

Proposition (Knop–Sahi, Okounkov '90s, Extra Vanishing Property)

$$h_\mu(\overline{\lambda}) = 0, \quad \textit{unless} \quad \lambda \supseteq \mu.$$

Write $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$, $1 \leqslant i \leqslant n$; $\lambda :\supset \mu$ if $\lambda \supseteq \mu$ and $|\lambda| = |\mu| + 1$.

Definition ((Adjacent) Binomial Coefficients)

$$b_{\lambda\mu} = \binom{\lambda}{\mu} = h_\mu(\overline{\lambda}), \quad a_{\lambda\mu} := \begin{cases} b_{\lambda\mu}, & \lambda :\supset \mu; \\ 0, & \text{otherwise}, \end{cases}$$

Binomial coefficients (for AJ) appear in the binomial formula:

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_\mu \binom{\lambda}{\mu}_\tau \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}$$

Some combinatorial formulas for adjacent binomial coefficients are given in [C–Sahi '24, Prop 4.3].

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For all four families of interpolation polynomials: AJ, AM, BJ, BM, they have the following vanishing, positivity and monotonicity properties.

- $\binom{n}{m} = 0$ unless $n \geqslant m$; $\binom{\lambda}{\mu} = 0$ unless $\lambda \supseteq \mu$ [Knop–Sahi, Okounkov, '90s];
- $\binom{n}{m} > 0$ when $n \geqslant m$; $\binom{\lambda}{\mu} > 0$ when $\lambda \supseteq \mu$ [Sahi '11, C–Sahi '24];
- $\binom{n}{k} \geqslant \binom{m}{k}$ when $n \geqslant m$; $\binom{\lambda}{\nu} \geqslant \binom{\mu}{\nu}$ when $\lambda \supseteq \mu$ [C–Sahi '24].

Here, $f \geqslant g$ means $f - g \in \mathbb{F}_{\geqslant 0}$, where $\mathbb{F}_{\geqslant 0}$ is some cone of positivity defined for each family. For AJ, $\mathbb{F}_{\geqslant 0} = \{ f/g \mid f, g \in \mathbb{Z}_{\geqslant 0}[\tau], g \neq 0 \}$; for AM, $\mathbb{F}_{\geqslant 0} = \{ f \in \mathbb{Q}(q,t) \mid f(q,t) > 0 \text{ when } q, t \in (0,1) \}$.

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Here, $f \geqslant g$ means $f - g \in \mathbb{F}_{\geqslant 0}$, where $\mathbb{F}_{\geqslant 0}$ is some cone of positivity defined for each family. For AJ, $\mathbb{F}_{\geqslant 0} = \{\, f/g \mid f, g \in \mathbb{Z}_{\geqslant 0}[\tau], g \neq 0 \,\}$; for AM, $\mathbb{F}_{\geqslant 0} = \{\, f \in \mathbb{Q}(q,t) \mid f(q,t) > 0 \text{ when } q, t \in (0,1) \,\}$.

For all four families of interpolation polynomials: AJ, AM, BJ, BM, they have the following vanishing, positivity and monotonicity properties.

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For Jack polynomials

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_{\nu\subseteq\lambda} \binom{\lambda}{\nu} \frac{P_\nu(x;\tau)}{P_\nu(\mathbf{1};\tau)}$$

Theorem (C–Sahi '24, Theorems 6.1 & 6.2)

TFAE:

- λ *contains* μ;
- $\dfrac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \dfrac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)}$ *is Jack positive;*
- $\dfrac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$ *is monomial positive;*
- $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is Schur positive;*
- $\dfrac{e_{\lambda'}(x+\mathbf{1})}{e_{\lambda'}(\mathbf{1})} - \dfrac{e_{\mu'}(x+\mathbf{1})}{e_{\mu'}(\mathbf{1})}$ *is elementary positive.*

Theorem (Cuttler–Greene–Skandera '11, Sra '16)

Let $|\lambda| = |\mu|$. *TFAE:*

- λ *majorizes* μ, *i.e.*, $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$, *for each* i;
- *(Muirhead's inequality)* $\frac{m_\lambda(x)}{m_\lambda(\mathbf{1})} - \frac{m_\mu(x)}{m_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n$;
- *(Newton's inequality)* $\frac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} - \frac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n$;
- *(Gantmacher's inequality)* $\frac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x)}{p_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n$;
- *(Sra's inequality)* $\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n$.

Theorem (Khare–Tao '18)

λ *weakly majorizes* μ *if and only if* $\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [1, \infty)^n$.

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Conjecture (C–Sahi '24, Conjecture 1)

- *(CGS Conjecture for Jack polynomials) Let* $|\lambda| = |\mu|$. λ *majorizes* μ *if and only if*

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n;$$

- *(KT Conjecture for Jack polynomials)* λ *weakly majorizes* μ *if and only if*

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Note: It suffices to prove that λ majorizes μ implies Eq. (9).

Temporarily, consider the family AJ and AM only.

Lemma (Sahi '11, Pieri Rule)

Let $\varepsilon_1 = (1, 0^{n-1}) = (1, 0, \ldots, 0)$ *and* $\mu \in \mathcal{P}_n$. *Then*

$$\left(h_{\varepsilon_1}(x) - h_{\varepsilon_1}(\overline{\mu})\right) \cdot h_\mu(x) = \sum_{\nu :\supset \mu} \left(h_{\varepsilon_1}(\overline{\nu}) - h_{\varepsilon_1}(\overline{\mu})\right) a_{\nu\mu} h_\nu(x). \tag{9}$$

Proof.

Compare the two sides at $\overline{\lambda}$ for $|\lambda| \leqslant |\mu| + 1$. □

Let $\left\|\overline{\lambda}\right\| = b_{\lambda\varepsilon_1} = h_{\varepsilon_1}(\overline{\lambda})$. Write $A = \left(a_{\lambda\mu}\right), B = \left(b_{\lambda\mu}\right), Z = \left(\|\overline{\mu}\| \, \delta_{\lambda\mu}\right)$.

Theorem (Sahi '11, Recursion Formula)

1. *The following recursions characterize $b_{\lambda\mu}$:*

$$(i)\ b_{\lambda\lambda} = 1; (ii)\ \left(\left\|\overline{\lambda}\right\| - \|\overline{\mu}\|\right) b_{\lambda\mu} = \sum_{\nu :\supset \mu} b_{\lambda\nu} \left(\|\overline{\nu}\| - \|\overline{\mu}\|\right) a_{\nu\mu}. \quad (10)$$

2. *The matrices A, B, Z satisfy the commutation relations:*

$$(i)\ [Z, B] = B[Z, A], \quad (ii)\ [Z, B^{-1}] = -[Z, A]B^{-1}. \quad (11)$$

Proof.

Evaluate the Pieri rule at $\overline{\lambda}$. □

Theorem (Sahi '11, Weighted Sum Formula for $b_{\lambda\mu}$)

Let $\lambda \supseteq \mu$*, and* $k := |\lambda| - |\mu|$.

$$b_{\lambda\mu} = \sum_{\zeta \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}(\zeta) \prod_{i=0}^{k-1} a_{\zeta_i \zeta_{i+1}}, \tag{12}$$

$$\mathrm{wt}(\zeta) := \prod_{i=0}^{k-1} \frac{\left\|\overline{\zeta_i}\right\| - \left\|\overline{\zeta_{i+1}}\right\|}{\left\|\overline{\zeta_0}\right\| - \left\|\overline{\zeta_{i+1}}\right\|}. \tag{13}$$

where the sum is over all the chains $\zeta = (\zeta_0, \ldots, \zeta_k)$ *with*

$$\lambda = \zeta_0 :\supset \zeta_1 :\supset \cdots :\supset \zeta_{k-1} :\supset \zeta_k = \mu,$$

(i.e., standard tableaux of skew shape λ/μ*).*

Main Results: Binomial Coefficients

Theorem (C–Sahi, Lemma 3.1, Theorem 3.2, Theorem A)

The aforementioned Pieri rule, recursion formula, and weighted sum formula hold for BJ, BM as well.

Theorem (C–Sahi, Theorem B, Positivity)

The binomial coefficient $b_{\lambda\mu} \in \mathbb{F}_{>0}$ if and only if $\lambda \supseteq \mu$.

Proof.

The weight $\mathrm{wt}(\zeta)$ are positive by definition and the adjacent binomial coefficients are positive by [C–Sahi '24, Prop 4.3], hence by the weighted sum formula, the binomial coefficients are also positive. □

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Theorem (C–Sahi, Theorem C, Monotonicity)

The difference $b_{\lambda\nu} - b_{\mu\nu} \in \mathbb{F}_{\geqslant 0}$ *if* $\lambda \supseteq \mu$.

Proof.

It suffices to show when $\lambda :\supset \mu$ by the telescoping series technique. For this, compare and examine the combinatorial formulas. □

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Definition (Littlewood–Richardson coefficients)

The **Littlewood–Richardson coefficients** are defined by the product expansion

$$h_\mu(x)h_\nu(x) = \sum_\lambda c_{\mu\nu}^\lambda h_\lambda(x). \tag{14}$$

Because the top degree terms of the interpolation polynomials are related to be ordinary ones, these $c_{\mu\nu}^\lambda$ generalize the usual LR coefficients for Jack/Macdonald polynomials.
They are good for induction or recursion purpose, as interpolation polynomials are inhomogeneous.

Fix ν, and write $C_\nu = \left(c_{\mu\nu}^{\lambda}\right)$, $D_\nu = \left(b_{\mu\nu}\delta_{\lambda\mu}\right)$.

Theorem (C–Sahi, Theorems 3.5 & 3.6, Recursion for $c_{\mu\nu}^{\lambda}$)

- *The following recursions characterize $c_{\mu\nu}^{\lambda}$:*

$$\begin{aligned}
&(i)\ c_{\lambda\mu}^{\lambda} = b_{\lambda\mu} \\
&(ii)\ \left(\left\|\overline{\lambda}\right\| - \|\overline{\mu}\|\right) c_{\mu\nu}^{\lambda} = \sum_{\zeta :\supset \mu} c_{\zeta\nu}^{\lambda} a_{\zeta\mu} \left(\left\|\overline{\zeta}\right\| - \|\overline{\mu}\|\right) \\
&\qquad - \sum_{\zeta \subset: \lambda} a_{\lambda\zeta} c_{\mu\nu}^{\zeta} \left(\left\|\overline{\lambda}\right\| - \left\|\overline{\zeta}\right\|\right).
\end{aligned} \tag{15}$$

- (i) $C = B^{-1}DB$; (ii) $[Z, C] = [C, [Z, A]]$.

Theorem (C–Sahi, Theorem D, Weighted Sum Formula for $c_{\mu\nu}^{\lambda}$)

$$c_{\mu\nu}^{\lambda} = \sum_{\zeta \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}_{\nu}^{\mathrm{LR}}(\zeta) \prod_{i=0}^{k-1} a_{\zeta_i \zeta_{i+1}}, \tag{16}$$

$$\mathrm{wt}_{\nu}^{\mathrm{LR}}(\zeta) := \sum_{j=0}^{k} \frac{\prod_{0 \leqslant i \leqslant k-1} \left(\left\|\overline{\zeta_i}\right\| - \left\|\overline{\zeta_{i+1}}\right\|\right)}{\prod_{\substack{0 \leqslant i \leqslant k \\ i \neq j}} \left(\left\|\overline{\zeta_j}\right\| - \left\|\overline{\zeta_i}\right\|\right)} b_{\zeta_j \nu}. \tag{17}$$

A special case: when $\lambda = \nu$, $c_{\mu\lambda}^{\lambda} = b_{\lambda\mu}$, and the weighted sum formula for LR coefficients *degenerates* to that for binomial coefficients.
Another special case: when $\lambda :\supset \mu$, $c_{\mu\nu}^{\lambda} = a_{\lambda\mu}(b_{\lambda\nu} - b_{\mu\nu})$.

Theorem (C–Sahi, Theorem E)

Assume $\lambda :\supset \mu$, then $c_{\mu\nu}^{\lambda} \in \mathbb{F}_{\geqslant 0}$ if $\lambda \supseteq \nu$.

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Thank you!

References I

Siddhartha Sahi. "The spectrum of certain invariant differential operators associated to a Hermitian symmetric space". In: *Lie theory and geometry*. Vol. 123. Progr. Math. Birkhäuser Boston, Boston, MA, 1994, pp. 569–576. DOI: 10.1007/978-1-4612-0261-5_21.

Friedrich Knop and Siddhartha Sahi. "Difference equations and symmetric polynomials defined by their zeros". In: *Internat. Math. Res. Notices* 10 (1996), pp. 473–486. DOI: 10.1155/S1073792896000311.

A. Okounkov and G. Olshanski. "Shifted Jack polynomials, binomial formula, and applications". In: *Mathematical research letters* 4.1 (1997), pp. 67–78. DOI: 10.4310/MRL.1997.v4.n1.a7.

References II

- Siddhartha Sahi. "Binomial coefficients and Littlewood–Richardson coefficients for interpolation polynomials and Macdonald polynomials". In: *Representation Theory and Mathematical Physics, Volume 557 of Contemp. Math.* (2011), pp. 359–369. DOI: 10.1090/conm/557/11039.
- Siddhartha Sahi. "Binomial Coefficients and Littlewood–Richardson Coefficients for Jack Polynomials". In: *International mathematics research notices* (2011). DOI: 10.1093/imrn/rnq126.
- Allison Cuttler, Curtis Greene, and Mark Skandera. "Inequalities for symmetric means". In: *European Journal of Combinatorics* 32.6 (Aug. 2011), pp. 745–761. DOI: 10.1016/j.ejc.2011.01.020.

References III

Suvrit Sra. “On inequalities for normalized Schur functions”. In: *European Journal of Combinatorics* 51 (Jan. 2016), pp. 492–494. DOI: 10.1016/j.ejc.2015.07.005.

Apoorva Khare and Terence Tao. “Schur polynomials, entrywise positivity preservers, and weak majorization”. In: *Séminaire Lotharingien de Combinatoire* 80B (2018).