Symmetric polynomials and interpolation polynomials

Hong Chen
hc813@math.rutgers.edu

Rutgers

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Binomial formula: the classical case

We have the classical Newton's binomial theorem.

Theorem

For non-negative integers m, n, and a variable x,

$$(x+1)^n = \sum_m \binom{n}{m} x^m$$

In fact, n could be any real number, and x is a real number in a neighborhood of 0.

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Partitions and symmetric polynomials

Fix $n \geqslant 1$, the number of variables.
A partition is a sequence $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

Denote by $\mathcal{P}_n$ the set of such partitions. The length $\ell(\lambda)$ is the number of non-zero parts, and the size is $|\lambda| = \sum \lambda_i$.

Many (any) interesting bases of the ring of symmetric polynomials are indexed by partitions, to name a few, the monomial $m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$, the power-sum $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, and the elementary $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where $p_r = m_{(r)} = \sum_i x_i^r$, $e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$.

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Schur, Jack and Macdonald polynomials

The most important basis is perhaps Schur polynomials s_λ.

$$s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})} = \det(e_{\lambda'_i-i+j}) = \sum_T \prod_{s\in\lambda} x_{T(s)}.$$

• For combinatorics, Schur polynomials are generating functions of Young tableaux.
• For rep theory, Schur polynomials correspond bijectively to all irreducible characters of the symmetric groups.

Jack $P_\lambda^{(\alpha)}(x)$ and Macdonald $P_\lambda(x;q,t)$ are generalizations of Schur and many other polynomials.
Schur: $\alpha = 1$ or $q = t$;
monomial: $\alpha = \infty$ or $t = 1$;
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Let $x = (x_1, \ldots, x_n)$ and $\mathbf{1} = (1^n) = (1, \ldots, 1)$. Okounkov and Olshanski ('97) prove the following for Jack polynomials:

$$(x+1)^n = \sum_m \binom{n}{m} x^m$$

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} = \sum_\mu \binom{\lambda}{\mu} \frac{P_\mu(x)}{P_\mu(\mathbf{1})}$$

$$\binom{n}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}\bigg|_{x=n}$$

$$\binom{\lambda}{\mu} = P^*_\mu(x)\bigg|_{x=\lambda}$$

These P^*_μ are called shifted Jack polynomials. A (un)shifted version is interpolation Jack polynomials [Sahi '94], [Knop–Sahi '96].

$x(x-1)\cdots(x-m+1)$ is inhomogeneous, with top deg part $= x^m$.
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Interpolation polynomials

Definition

The unital interpolation polynomial, denoted by h_μ, is the unique symmetric function that satisfies the following interpolation condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad \forall \lambda \in \mathcal{P}_n,\ |\lambda| \leqslant |\mu|, \tag{1}$$

$$\deg h_\mu = |\mu| \tag{2}$$

where $\overline{\lambda}_i = \lambda_i + (n-i)/\alpha$.

Let $n = 2$, $\mu = (3,2)$, $h_\mu^{\text{monic}} = x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 + x_2 - 1/\alpha - 4)$.
The defining condition involves the following 12 partitions:

$$(0,0),(1,0),(1,1),(2,0),(2,1),(3,0),(2,2),(3,1),(4,0),(3,2),(4,1),(5,0).$$

We have $\overline{(\lambda_1,\lambda_2)} = (\lambda_1 + 1/\alpha, \lambda_2)$. Because of the factorizations, one can easily see that h_μ vanishes at all but $\overline{(3,2)}$.
Moreover, h_μ also vanishes at $\overline{(m,0)}$ and $\overline{(m-1,1)}$ $\forall m \geqslant 6$.
Warning: In most cases, no complete factorization!

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Vanishing, positivity, monotonicity

Define a partial order on partitions: $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$ for each i.

- $\binom{n}{m} = 0$ unless $n \geqslant m$; $\binom{\lambda}{\mu} = 0$ unless $\lambda \supseteq \mu$ [Knop–Sahi, Okounkov '90s];
- $\binom{n}{m} > 0$ when $n \geqslant m$; $\binom{\lambda}{\mu} > 0$ when $\lambda \supseteq \mu$ [Sahi '11, C–Sahi '24];
- $\binom{n}{k} \geqslant \binom{m}{k}$ when $n \geqslant m$; $\binom{\lambda}{\nu} \geqslant \binom{\mu}{\nu}$ when $\lambda \supseteq \mu$ [C–Sahi '24].

Moreover, there are interpolation Macdonald polynomials; as well as interpolation Jack and Macdonald polynomials of type BC.
We prove the positivity and the monotonicity for binomial coefficients associated to these interpolation polynomials (Theorems B and C).

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Application

For Jack polynomials

$$P_\lambda(x+\mathbf{1}) = \sum_\nu \binom{\lambda}{\nu} P_\nu(x)$$

Theorem (C–Sahi '24, Theorem F)

TFAE:

- λ *contains* μ;
- $\dfrac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \dfrac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})}$ *is* ***Jack positive;***
- $\dfrac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$ *is* ***monomial positive;***
- $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is* ***Schur positive;***
- $\dfrac{e_{\lambda'}(x+\mathbf{1})}{e_{\lambda'}(\mathbf{1})} - \dfrac{e_{\mu'}(x+\mathbf{1})}{e_{\mu'}(\mathbf{1})}$ *is* ***elementary positive.***

Related results

Theorem (Cuttler–Greene–Skandera '11, Sra '16)

Let $|\lambda| = |\mu|$. *TFAE:*

- λ *majorizes* μ, *i.e.,* $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$, *for each* i;
- *(Muirhead's inequality)* $\dfrac{m_\lambda(x)}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x)}{m_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
- *(Newton's inequality)* $\dfrac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} - \dfrac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
- *(Gantmacher's inequality)* $\dfrac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \dfrac{p_\mu(x)}{p_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
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Theorem (Khare–Tao '18)

λ *weakly majorizes* μ *if and only if* $\dfrac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$.

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Conjecture

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- *(CGS Conjecture for Jack polynomials) Let* $|\lambda| = |\mu|$. λ *majorizes* μ *if and only if*

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n; \tag{3}$$

- *(KT Conjecture for Jack polynomials)* λ *weakly majorizes* μ *if and only if*

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Note: It suffices to prove that λ majorizes μ implies Eq. (3).