Interpolation Polynomials, Binomial Coefficients, and Symmetric Function Inequalities

Hong Chen (Joint with Siddhartha Sahi)
arXiv:2403.02490
Rutgers

Abstract

We study the binomial coefficients and Littlewood–Richardson coefficients associated to interpolation polynomials. We give a weighted sum formula and prove that such binomial coefficients are positive and monotone. As an application, we prove various inequalities on Jack polynomials and their specializations, monomial, Schur and elementary symmetric polynomials, generalizing similar inequalities due to Cuttler–Greene–Skandera, Sra and Khare–Tao.

Motivations and Preliminaries

The following Newton's binomial formula is well-known:

$$(x+1)^n = \sum_m \binom{n}{m} x^m. \tag{1}$$

The **binomial coefficient** $\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!}$ satisfies many simple properties:

- (Polynomiality) $(\binom{n}{m})_n$ is a polynomial;
- (Positivity) $\binom{n}{m} > 0$ if $n \geqslant m$;
- (Vanishing) $\binom{n}{m} = 0$ unless $n \geqslant m$;
- (Monotonicity) $\binom{n}{k} \geqslant \binom{m}{k}$ if $n \geqslant m$.

One natural question is to generalize these into symmetric polynomials in n variables. Symmetric polynomials are usually indexed by partitions. A **partition** of length at most n is an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The monomial m_λ, elementary e_λ, Schur s_λ and power-sum p_λ are the most basic examples of symmetric polynomials. **Jack polynomials** are one-parameter generalizations of the first three: they depend on a parameter τ $(= 1/\alpha)$, and reduce to the monomial, Schur and transposed elementary when $\tau = 0, 1, \infty$. **Macdonald polynomials** are two-parameter generalizations: they depend on two parameters q, t, and reduce to the monomial, Schur and transposed elementary when $t = 1, q = t, q = 1$ (and many other interesting families).

Okounkov–Olshanski generalized the binomial theorem to Jack polynomials:

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_\mu \binom{\lambda}{\mu}_\tau \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}, \tag{2}$$

where $x = (x_1, \dots, x_n)$, $\mathbf{1} = (1, \dots, 1)$, P_λ is the monic Jack polynomial, and $\binom{\lambda}{\mu}_\tau$ is the **generalized binomial coefficient**. Okounkov–Olshanski showed the **polynomiality** of generalized binomial coefficients: they are evaluations of the so-called **interpolation Jack polynomials** (also known as shifted Jack polynomials), first defined by Knop–Sahi. Okounkov also found a binomial formula for Macdonald polynomials, whose binomial coefficients are given by **interpolation Macdonald polynomials**.

Let $\mathbb{F}$ be the field of rational functions in the corresponding parameters, $\mathbb{F} = \mathbb{Q}(\tau), \mathbb{Q}(q,t)$. The **interpolation polynomials**, of *unital* normalization, can be uniformly defined as the unique symmetric polynomial over $\mathbb{F}$ satisfying the following interpolation condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad |\lambda| \leqslant |\mu|; \qquad \deg h_\mu = |\mu|, \tag{3}$$

where $|\lambda| = \sum \lambda_i$ is the size of λ and the **shifting** is given by $\overline{\lambda}_i = \lambda_i + (n-i)\tau, q^{\lambda_i} t^{n-i}$ for Jack and Macdonald, respectively.

For example, when $n = 2$ and $\mu = (3,2)$, the *monic* interpolation Jack polynomial is

$$h^{\text{monic}}_{(3,2)}(x_1, x_2) = x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 + x_2 - \tau - 4).$$

One can easily verify that $h^{\text{monic}}_{(3,2)}(\overline{(3,2)}) = h^{\text{monic}}_{(3,2)}(3+\tau, 2) \neq 0$ and $h^{\text{monic}}_{(3,2)}$ vanishes at $\overline{(2,2)} = (2+\tau, 2)$, $\overline{(m,0)} = (m+\tau, 0)$ and $\overline{(m-1,1)} = (m-1+\tau, 1)$, more points than required in the definition. In general, such phenomenon is known as the **extra vanishing property**:

$$\binom{\lambda}{\mu} := h_\mu(\overline{\lambda}) = 0 \quad \text{unless} \quad \lambda \supseteq \mu. \tag{4}$$

These evaluations are called the (generalized) **binomial coefficients**. When $\mu = (1)$ or $|\lambda| = |\mu| + 1$, they can be computed by some combinatorial formulas in Proposition 4.3.

The interpolation **Littlewood–Richardson coefficients** are defined by the product expansion:

$$h_\mu(x) h_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} h_\lambda(x). \tag{5}$$

Some Inequalities

Recall that for partitions λ and μ of length at most n, we say λ **weakly majorizes** (or, **weakly dominates**) μ, if $\sum_{i=1}^r \lambda_i \geqslant \sum_{i=1}^r \mu_i$, for $1 \leqslant r \leqslant n$; if, in addition, $|\lambda| = |\mu|$, we say λ **majorizes** (or, **dominates**) μ.

In the work of Muirhead, Cuttler–Greene–Skandera and Sra, they showed that majorization can be characterized by the following inequalities: assume $|\lambda| = |\mu|$,

$$\begin{aligned}
\lambda \text{ majorizes } \mu \iff & \frac{m_\lambda}{m_\lambda(\mathbf{1})} - \frac{m_\mu}{m_\mu(\mathbf{1})} \geqslant 0 \iff \frac{e'_\lambda}{e'_\lambda(\mathbf{1})} - \frac{e'_\mu}{e'_\mu(\mathbf{1})} \geqslant 0 \\
\iff & \frac{p_\lambda}{p_\lambda(\mathbf{1})} - \frac{p_\mu}{p_\mu(\mathbf{1})} \geqslant 0 \iff \frac{s_\lambda}{s_\lambda(\mathbf{1})} - \frac{s_\mu}{s_\mu(\mathbf{1})} \geqslant 0,
\end{aligned}$$

where $m_\lambda, e_\lambda, p_\lambda, s_\lambda$ are monomial, elementary, power-sum and Schur polynomials and a polynomial $f \geqslant 0$ means that when evaluated at the positive orthant $x \in [0,\infty)^n$, $f(x) \geqslant 0$.

Khare–Tao showed a similar characterization for weak majorization:

$$\lambda \text{ weakly majorizes } \mu \iff \frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n.$$

Our Work

We prove that the binomial coefficients and the interpolation Littlewood–Richardson coefficients satisfy some similar properties as the classical ones.

Theorem 1 (Positivity and Monotonicity)

View the parameter as real numbers satisfying: $\tau > 0$, and $0 < q, t < 1$.

- *(Positivity) $\binom{\lambda}{\mu} > 0$ if and only if $\lambda \supseteq \mu$;*
- *(Monotonicity) $\binom{\lambda}{\nu} \geqslant \binom{\mu}{\nu}$ if $\lambda \supseteq \mu$;*
- *(Positivity) $c^\lambda_{\mu\nu} \geqslant 0$ if $\lambda = \mu$ or $|\lambda| = |\mu| + 1$.*

□

As an application of the monotonicity, we have the following characterization:

Theorem 2 (Characterization of Containment)

Let λ and μ be partitions of length at most n. Then λ contains μ if and only if one of the following holds:

- *$\frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ is Schur positive;*
- *$\frac{e_\lambda(x+\mathbf{1})}{e_\lambda(\mathbf{1})} - \frac{e_\mu(x+\mathbf{1})}{e_\mu(\mathbf{1})}$ is elementary positive;*
- *$\frac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \frac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$ is monomial positive;*
- *$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)}$ is Jack positive,*

where the Jack positivity is over $\mathbb{F}_{\geqslant 0} := \{f(\tau) \in \mathbb{Q}(\tau) \mid f(\tau_0) \geqslant 0, \forall \tau_0 \in [0,\infty]\}$. □

For example, write $S_\lambda(x) = \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}$ and $\widetilde{S}_\lambda(x) = S_\lambda(x+\mathbf{1})$, and similarly for M and $\widetilde{M}$, E and $\widetilde{E}$, P^* and $\widetilde{P}^*$, then

$$\begin{aligned}
\widetilde{S}_{(3,1)} - \widetilde{S}_{(2)} &= S_{(3,1)} + \tfrac{4}{3} S_{(3)} + \tfrac{8}{3} S_{(2,1)} + 3S_{(2)} + 2S_{(1,1)} + 2S_{(1)}; \\
\widetilde{M}_{(3,1)} - \widetilde{M}_{(2)} &= M_{(3,1)} + M_{(3)} + 3M_{(2,1)} + 2M_{(2)} + 3M_{(1,1)} + 2M_{(1)}; \\
\widetilde{E}_{(2,1,1)} - \widetilde{E}_{(1,1)} &= E_{(2,1,1)} + 2E_{(1,1,1)} + 2E_{(2,1)} + 4E_{(1,1)} + E_{(2)} + 2E_{(1)}; \\
\widetilde{P}^*_{(3,1)} - \widetilde{P}^*_{(2)} &= P^*_{(3,1)} + \tfrac{2\tau+2}{\tau+2} P^*_{(3)} + \tfrac{2\tau+6}{\tau+2} P^*_{(2,1)} + \tfrac{4\tau+2}{\tau+1} P^*_{(2)} + \tfrac{\tau+3}{\tau+1} P^*_{(1,1)} + 2P^*_{(1)}.
\end{aligned}$$

The positivity of the binomial coefficients are proved using the following weighted sum formulas. Denote by $\mathfrak{C}_{\lambda\mu}$ the set of paths $\boldsymbol{\kappa} = (\kappa(0), \dots, \kappa(k))$ from λ to μ, i.e.,

$$\lambda = \kappa(0) \supseteq \kappa(1) \supseteq \cdots \supseteq \kappa(k-1) \supseteq \kappa(k) = \mu,$$

such that $|\kappa(i)| - |\kappa(i+1)| = 1$.

Theorem 3 (Weighted Sum Formulas)

The binomial coefficients $\binom{\lambda}{\mu}$ and the interpolation LR coefficients $c^\lambda_{\mu\nu}$ admit the following weighted sum formulas:

$$\binom{\lambda}{\mu} = \sum_{\boldsymbol{\kappa} \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}(\boldsymbol{\kappa}) \binom{\lambda}{\kappa(1)} \cdots \binom{\kappa(k-1)}{\mu}, \tag{6}$$

$$c^\lambda_{\mu\nu} = \sum_{\boldsymbol{\kappa} \in \mathfrak{C}_{\lambda\mu}} \mathrm{wt}^{\mathrm{LR}}_\nu(\boldsymbol{\kappa}) \binom{\lambda}{\kappa(1)} \cdots \binom{\kappa(k-1)}{\mu}, \tag{7}$$

*where the weights $\mathrm{wt}(\boldsymbol{\kappa})$ and $\mathrm{wt}^{\mathrm{LR}}_\nu(\boldsymbol{\kappa})$ are given by Eqs. (3.8) and (3.20) of the paper; and each of the **adjacent binomial coefficients** $\binom{\kappa(i)}{\kappa(i+1)}$ can be explicitly computed via Proposition 4.3. In particular, $c^\lambda_{\mu\nu} = \binom{\lambda}{\nu}$ if $\lambda = \mu$ and $c^\lambda_{\mu\nu} = \binom{\lambda}{\mu}\left(\binom{\lambda}{\nu} - \binom{\mu}{\nu}\right)$ if $|\lambda| = |\mu| + 1$.* □

The formula for binomial coefficients are first discovered in 2011 by Sahi and we generalize it to type BC analogues of interpolation polynomials; the formula for LR coefficients are new in all cases, interpolation Jack and Macdonald polynomials of types A and BC.

We end with a conjecture that generalizes the inequalities of Cuttler–Greene–Skandera and Khare–Tao to Jack polynomials:

Conjecture

Let λ and μ be partitions of length at most n, P_λ be the monic Jack polynomials, and let $\mathbb{F}^{\mathbb{R}}_{\geqslant 0} := \{ f(\tau) \in \mathbb{R}(\tau) \mid f(\tau_0) \geqslant 0, \forall \tau_0 \in [0,\infty] \}$.

- *(CGS Conjecture for Jack polynomials) Assume $|\lambda| = |\mu|$. λ majorizes μ if and only if*

$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{8}$$

- *(KT Conjecture for Jack polynomials) λ weakly majorizes μ if and only if*

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{9}$$

Note that the CGS conjecture, together with Theorem 2, implies the KT conjecture; and that the "if" direction of it is easily seen to be true by some degree consideration.

References

[CGS11] Allison Cuttler, Curtis Greene, and Mark Skandera. Inequalities for symmetric means. *European J. Combin.*, 32(6):745–761, 2011.

[KT18] Apoorva Khare and Terence Tao. Schur polynomials, entrywise positivity preservers, and weak majorization. *Sém. Lothar. Combin.*, 80B:Art. 14, 12, 2018.

[Sah11a] Siddhartha Sahi. Binomial coefficients and Littlewood-Richardson coefficients for interpolation polynomials and Macdonald polynomials. In *Representation theory and mathematical physics*, volume 557 of *Contemp. Math.*, pages 359–369. Amer. Math. Soc., Providence, RI, 2011.

[Sah11b] Siddhartha Sahi. Binomial coefficients and Littlewood-Richardson coefficients for Jack polynomials. *Int. Math. Res. Not. IMRN*, (7):1597–1612, 2011.

[Sra16] Suvrit Sra. On inequalities for normalized Schur functions. *European J. Combin.*, 51:492–494, 2016.