Binomial Coefficients for Interpolation Polynomials

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For non-negative integers m, n, and an indeterminate x,

$$(x+1)^n = \sum_m \binom{n}{m} x^m.$$

Newton: n could be any real number, and x is a real number in a neighborhood of 0.
Cauchy: q-binomial formula

$$\prod_{i=1}^{n}(1+xq^i) = \sum_m \binom{n}{m}_q q^{m(m+1)/2} x^m.$$

Question

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How to generalize this to symmetric polynomials?

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n) \in \mathbb{Z}^n_{\geqslant 0}$ be an integer partition. Schur polynomial in $x = (x_1, \ldots, x_n)$ is defined by

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$$

In 1978, Lascoux (see [Macdonald, P.47]) showed that

$$s_\lambda(x + \mathbf{1}) = \sum_{\mu \subseteq \lambda} d_{\lambda\mu} s_\mu(x),$$

where $\mathbf{1} = (1, \ldots, 1)$ and

$$d_{\lambda\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1 \leqslant i,j \leqslant n}.$$

Jack polynomial $P_\lambda(x;\tau)$ is a deformation of Schur, depending on a parameter $\tau(=1/\alpha)$:
$\tau=1$: Schur; $\tau=0$: monomial; $\tau=\infty$: transposed elementary;
$\tau=\frac{1}{2},2$: Zonal.
In the '90s, Lassalle, Kaneko, Okounkov–Olshanski studied the Jack case.

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)}=\sum_\mu \binom{\lambda}{\mu}_\tau \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)},$$

where the binomial coefficient is the evaluation of the **interpolation Jack polynomial** h_μ:

$$\binom{\lambda}{\mu}_\tau = h_\mu(\overline{\lambda};\tau)$$

Macdonald polynomial $P_\lambda(x;q,t)$ depends on two parameters q,t:
$q=t$: Schur; $t=1$: monomial; $q=1$: transposed elementary;
$q=0$: Hall–Littlewood; $t=0$: q-Whittaker
In 1998, Okounkov and Lassalle proved:

$$\frac{P_\lambda(x;q,t)}{P_\lambda(t^\delta;q,t)} = \sum_\mu \binom{\lambda}{\mu}_{q,t} \frac{h_\mu^{\text{monic}}(x;\tau)}{P_\mu(t^\delta;\tau)},$$

where the binomial coefficient is the evaluation of the **interpolation Macdonald polynomial** h_μ:

$$\binom{\lambda}{\mu}_{q,t} = h_\mu(\overline{\lambda};q,t)$$

Definition

The *unital interpolation polynomial*, denoted by h_μ, is the unique symmetric polynomial that satisfies the following interpolation condition and degree condition:

$$h_\mu(\overline{\lambda}) = \delta_{\lambda\mu}, \quad |\lambda| \leqslant |\mu|,$$
$$\deg h_\mu = |\mu|,$$

where $\overline{\lambda}_i = \lambda_i + (n-i)\tau$ in the Jack case, and $\overline{\lambda}_i = q^{\lambda_i} t^{n-i}$ in the Macdonald case.

This normalization is called *unital* as $\binom{\mu}{\mu} = h_\mu(\overline{\mu}) = 1$.

The *monic* normalization h_μ^{monic} is defined so that the coefficient of m_μ is 1. $h_\mu^{\text{monic}} = P_\mu +$ lower degree terms.

The above is type A interpolation polynomials, introduced by Knop–Sahi in the '90s. Okounkov also introduced a type BC analogue, defined by setting $\overline{\lambda}_i = \lambda_i + (n-i)\tau + \alpha$ and $aq^{\lambda_i} t^{n-i}$.

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Okounkov–Olshanski proved the following combinatorial formulas:

$$P_\lambda^{\mathrm{monic,J}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},AJ}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau \right) \right),$$

$$h_\lambda^{\mathrm{monic},BJ}(x;\tau,\alpha) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x^2_{T(s)} - \left(a'_\lambda(s) + (n - T(s) - l'_\lambda(s))\tau + \alpha \right)^2 \right),$$

$$P_\lambda^{\mathrm{monic,M}}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)},$$

$$h_\lambda^{\mathrm{monic},AM}(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \left(x_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} \right),$$

$$h_\lambda^{\mathrm{monic},BM}(x;q,t,a) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} \Bigg(x_{T(s)} + x^{-1}_{T(s)} - q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a - \left(q^{a'_\lambda(s)} t^{n-T(s)-l'_\lambda(s)} a \right)^{-1} \Bigg)$$

where all the sums are over column-strict semi-standard reverse tableaux $T : \lambda \to [n]$, i.e., weakly decreasing along the rows and strictly decreasing along the columns.

Let $n=2$, $\mu=(3,2)$, there are two such tableaux: $\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 1&1\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|}\hline 2&2&2\\\hline 1&1\\\cline{1-2}\end{array}$.

Hence $s_\mu = P^{\mathrm{J}}_\mu = P^{\mathrm{M}}_\mu = m_\mu = x_1^3x_2^2 + x_1^2x_2^3$,

$$\begin{aligned} h_\mu^{\mathrm{monic},AJ} &= x_2x_1(x_2-1)(x_1-1)(x_1-2-\tau)\\ &\quad + x_2x_1(x_2-1)(x_1-1)(x_2-2)\\ &= x_1x_2(x_1-1)(x_2-1)(x_1+x_2-\tau-4)\end{aligned}$$

$$\begin{aligned} h_\mu^{\mathrm{monic},AM} &= (x_2-1)(x_1-1)(x_2-q)(x_1-q)(x_1-q^2t)\\ &\quad + (x_2-1)(x_1-1)(x_2-q)(x_1-q)(x_2-q^2)\\ &= (x_1-1)(x_2-1)(x_1-q)(x_2-q)(x_1+x_2-q^2t-q^2)\end{aligned}$$

The defining condition involves the following 12 partitions:

$$(0,0),(1,0),(1,1),(2,0),(2,1),(3,0),(2,2),(3,1),(4,0),(3,2),(4,1),(5,0).$$

Note that $\overline{(\lambda_1,\lambda_2)} = (\underline{\lambda_1+\tau},\lambda_2),(q^{\lambda_1}t,q^{\lambda_2})$. One can easily see that h_μ vanishes at all but $\overline{(3,2)}$.

Moreover, h_μ also vanishes at $\overline{(m,0)}$ and $\overline{(m-1,1)}$, $\forall m\geqslant 6$.

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Note that $\overline{(\lambda_1,\lambda_2)} = (\lambda_1+\tau,\lambda_2),(q^{\lambda_1}t,q^{\lambda_2})$. One can easily see that h_μ vanishes at all but $\overline{(3,2)}$.

Moreover, h_μ also vanishes at $\overline{(m,0)}$ and $\overline{(m-1,1)}$, $\forall m \geqslant 6$.

Proposition (Knop–Sahi, Okounkov '90s, Extra Vanishing Property)

$$\binom{\lambda}{\mu} := h_\mu(\overline{\lambda}) = 0, \quad \textit{unless} \quad \lambda \supseteq \mu. \qquad \binom{m}{n} = 0, \quad \textit{unless} \quad m \geqslant n.$$

Theorem (Sahi '11, C–Sahi '24)

The binomial coefficient is positive and monotone:

$$\binom{\lambda}{\mu} \in \mathbb{F}_{>0} \iff \lambda \supseteq \mu. \qquad \binom{m}{n} > 0 \iff m \geqslant n.$$

$$\binom{\lambda}{\nu} - \binom{\mu}{\nu} \in \mathbb{F}_{\geqslant 0} \quad \textit{if} \quad \lambda \supseteq \mu. \qquad \binom{m}{k} - \binom{n}{k} > 0 \quad \textit{if} \quad m > n.$$

Here, $\mathbb{F}_{\geqslant 0}$ and $\mathbb{F}_{>0}$ is the cone of positivity (defined later).

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Here, $\mathbb{F}_{\geqslant 0}$ *and* $\mathbb{F}_{>0}$ *is the cone of positivity (defined later).*

For Jack polynomials

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_{\nu\subseteq\lambda} \binom{\lambda}{\nu} \frac{P_\nu(x;\tau)}{P_\nu(\mathbf{1};\tau)}$$

Theorem (C–Sahi '24)

TFAE:

- λ *contains* μ, *i.e.,* $\lambda_i \geqslant \mu_i$ *for each* i;
- $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$ *is Schur positive;*
- $\dfrac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$ *is monomial positive;*
- $\dfrac{e_\lambda(x+\mathbf{1})}{e_\lambda(\mathbf{1})} - \dfrac{e_\mu(x+\mathbf{1})}{e_\mu(\mathbf{1})}$ *is elementary positive.*
- $\dfrac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \dfrac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)}$ *is Jack positive over* $\mathbb{F}_{\geqslant 0}$;

Write $S_\lambda(x) = \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})}$ and $\widetilde{S}_\lambda(x) = S_\lambda(x+\mathbf{1})$, and similarly for M and $\widetilde{M}$, E and $\widetilde{E}$, P^* and $\widetilde{P}^*$, then

$$\begin{aligned}
\widetilde{S}_{\ydiagram{3,1}} - \widetilde{S}_{\ydiagram{2}} &= S_{\ydiagram{3,1}} + \tfrac{4}{3} S_{\ydiagram{3}} + \tfrac{8}{3} S_{\ydiagram{2,1}} + 3S_{\ydiagram{2}} + 2S_{\ydiagram{1,1}} + 2S_{\ydiagram{1}}\,;\\
\widetilde{M}_{\ydiagram{3,1}} - \widetilde{M}_{\ydiagram{2}} &= M_{\ydiagram{3,1}} + M_{\ydiagram{3}} + 3M_{\ydiagram{2,1}} + 2M_{\ydiagram{2}} + 3M_{\ydiagram{1,1}} + 2M_{\ydiagram{1}}\,;\\
\widetilde{E}_{\ydiagram{2,1,1}} - \widetilde{E}_{\ydiagram{1,1}} &= E_{\ydiagram{2,1,1}} + 2E_{\ydiagram{1,1,1}} + 2E_{\ydiagram{2,1}} + 4E_{\ydiagram{1,1}} + E_{\ydiagram{2}} + 2E_{\ydiagram{1}}\,;\\
\widetilde{P}^*_{\ydiagram{3,1}} - \widetilde{P}^*_{\ydiagram{2}} &= P^*_{\ydiagram{3,1}} + \tfrac{2\tau+2}{\tau+2} P^*_{\ydiagram{3}} + \tfrac{2\tau+6}{\tau+2} P^*_{\ydiagram{2,1}} + \tfrac{4\tau+2}{\tau+1} P^*_{\ydiagram{2}} + \tfrac{\tau+3}{\tau+1} P^*_{\ydiagram{1,1}} + 2P^*_{\ydiagram{1}}\,.
\end{aligned}$$

Recall that when $\tau = 1, 0, \infty$, Jack specializes to Schur, monomial and transposed elementary respectively.

$\mathbb{F} = \mathbb{Q}(\tau)$, let $\mathbb{F}_{\geqslant 0} := \{\, f/g \mid f, g \in \mathbb{Z}_{\geqslant 0}[\tau], g \neq 0 \,\}$ and $\mathbb{F}_{>0} := \mathbb{F}_{\geqslant 0} \setminus \{0\}$. In particular, $f(\tau_0) > 0$ for $f \in \mathbb{F}_{>0}$ and $\tau_0 \in (0, \infty)$.

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$$
\begin{array}{ccccccccccccccc}
\widetilde{S}_{(3,1)} & - & \widetilde{S}_{(2)} & = & S_{(3,1)} & + & \frac{4}{3}S_{(3)} & + & \frac{8}{3}S_{(2,1)} & + & 3S_{(2)} & + & 2S_{(1,1)} & + & 2S_{(1)}\,; \\
\widetilde{M}_{(3,1)} & - & \widetilde{M}_{(2)} & = & M_{(3,1)} & + & M_{(3)} & + & 3M_{(2,1)} & + & 2M_{(2)} & + & 3M_{(1,1)} & + & 2M_{(1)}\,; \\
\widetilde{E}_{(2,1,1)} & - & \widetilde{E}_{(1,1)} & = & E_{(2,1,1)} & + & 2E_{(1,1,1)} & + & 2E_{(2,1)} & + & 4E_{(1,1)} & + & E_{(2)} & + & 2E_{(1)}\,; \\
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Theorem (Cuttler–Greene–Skandera '11, Sra '16)

Let $|\lambda| = |\mu|$. *TFAE:*

- λ *dominates* μ, *i.e.*, $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$, *for each* i;
- *(Muirhead's inequality)* $\dfrac{m_\lambda(x)}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x)}{m_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
- *(Newton's inequality)* $\dfrac{e_{\lambda'}(x)}{e_{\lambda'}(\mathbf{1})} - \dfrac{e_{\mu'}(x)}{e_{\mu'}(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
- *(Gantmacher's inequality)* $\dfrac{p_\lambda(x)}{p_\lambda(\mathbf{1})} - \dfrac{p_\mu(x)}{p_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$;
- *(Sra's inequality)* $\dfrac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x)}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$.

Theorem (Khare–Tao '18)

λ *weakly dominates* μ $\iff$ $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n$.

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Theorem (Khare–Tao '18)

$$\lambda \text{ weakly dominates } \mu \iff \frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0, \infty)^n.$$

In two variables,

$$
\begin{aligned}
S_{\ydiagram{4}} \quad &- \quad S_{\ydiagram{3,1}} &= \quad & \frac{1}{15}(x_1 - x_2)^2(3x_1^2 + 4x_1x_2 + 3x_2^2) \\
S_{\ydiagram{3,1}} \quad &- \quad S_{\ydiagram{2,2}} &= \quad & \frac{1}{3}(x_1 - x_2)^2 x_1 x_2 \\
P^*_{\ydiagram{4}} \quad &- \quad P^*_{\ydiagram{3,1}} &= \quad & \frac{(\tau+3)(x_1 - x_2)^2}{4(2\tau+1)(2\tau+3)}\Big(\tau(x_1 + x_2)^2 + 2(x_1^2 + x_1x_2 + x_2^2)\Big) \\
P^*_{\ydiagram{3,1}} \quad &- \quad P^*_{\ydiagram{2,2}} &= \quad & \frac{(\tau+1)}{2(2\tau+1)}(x_1 - x_2)^2 x_1 x_2
\end{aligned}
$$

Conjecture (C–Sahi '24)

- *(CGS Conjecture for Jack polynomials) Let* $|\lambda| = |\mu|$. λ *dominates* μ *if and only if*

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x)}{P_\mu(\mathbf{1})} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0, \infty)^n;$$

- *(KT Conjecture for Jack polynomials)* λ *weakly dominates* μ *if and only if*

$$\frac{P_\lambda(x+\mathbf{1})}{P_\lambda(\mathbf{1})} - \frac{P_\mu(x+\mathbf{1})}{P_\mu(\mathbf{1})} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0, \infty)^n.$$

Here, $\mathbb{F}^{\mathbb{R}}_{\geqslant 0} \coloneqq \{\, f/g \mid f \in \mathbb{R}_{\geqslant 0}[\tau], g \in \mathbb{Z}_{\geqslant 0}[\tau], g \neq 0 \,\}$

Note that the CGS conjecture, together with our theorem, implies the KT conjecture; also, the "if" direction of the CGS conjecture is easily seen to be true by some degree consideration.

Thank you!

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slides: https://sites.math.rutgers.edu/~hc813/