Binomial formulas for symmetric polynomials

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Research Bonanza

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The classical case

We have the classical Newton's binomial theorem.

Theorem

For non-negative integers m, n, and a variable x,

$$(x+1)^n = \sum_m \binom{n}{m} x^m$$

In fact, n could be any real number, and x is a real number in a neighborhood of 0.

Question

How to generalize this to symmetric polynomials, that is, multivariate polynomials whose variables are symmetric?

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Partitions and Schur functions

A partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0.$$

Many (any) interesting bases of the ring of symmetric polynomials are indexed by partitions, to name a few, the monomial m_λ, the power-sum p_λ, the elementary e_λ, the complete h_λ.

The most important and interesting are perhaps Schur polynomials s_λ.
• For rep theory, Schur polynomials correspond bijectively to all irreducible characters of the symmetric groups.
• For combinatorics, $s_\lambda(x) = \sum_T x^{\mathrm{wt}(T)}$ is the generating function of Young tableaux.

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The binomial formula for Schur

Let $x = (x_1, \ldots, x_N)$ and $\mathbf{1} = (1^N) = (1, \ldots, 1)$, and $S_\lambda(x) = s_\lambda(x)/s_\lambda(\mathbf{1})$ is the normalized Schur.
Okounkov and Olshanski prove the following:

$$(x+1)^n = \sum_m \binom{n}{m} x^m$$

$$S_\lambda(x+\mathbf{1}) = \sum_\mu \binom{\lambda}{\mu} S_\mu(x)$$

$$\binom{n}{m} = \left.\frac{x(x-1)\cdots(x-m+1)}{m!}\right|_{x=n}$$

$$\binom{\lambda}{\mu} = \left. S^*_\mu(x)\right|_{x=\lambda}$$

These S^*_μ are called interpolation Schur (also known as shifted Schur, factorial Schur).

$x(x-1)\cdots(x-m+1)$ is inhomo. of degree m, top deg part $= x^m$.
S^*_μ is inhomo. of degree $\deg(S_\mu) = |\mu|$, top deg part $=$ multiple of S_μ.

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Vanishing, positivity, monotonicity and our work

Define a partial order on partitions: $\lambda \supseteq \mu$ if $\lambda_i \geqslant \mu_i$ for each i.

- $\binom{n}{m} = 0$ unless $n \geqslant m$; $\binom{\lambda}{\mu} = 0$ unless $\lambda \supseteq \mu$.
- $\binom{n}{m} > 0$ when $n \geqslant m$; $\binom{\lambda}{\mu} > 0$ when $\lambda \supseteq \mu$.
- $\binom{n}{m} \geqslant \binom{n'}{m}$ when $n \geqslant n'$; $\binom{\lambda}{\mu} \geqslant \binom{\lambda'}{\mu}$ when $\lambda \supseteq \lambda'$.

The vanishing and positivity are known. We prove the monotonicity.

There are Jack and Macdonald generalization of interpolation polynomials, of types A and BC.
We prove the positivity and the monotonicity for binomial coefficients associated to these interpolation polynomials (Theorems B and C).

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Application

$$S_\lambda(x+\mathbf{1}) = \sum_\nu \binom{\lambda}{\nu} S_\nu(x)$$

$$P_\lambda(x+\mathbf{1}) = \sum_\nu \binom{\lambda}{\nu} P_\nu(x)$$

Theorem (C–Sahi, Theorem F)

λ contains μ if and only if $S_\lambda(x+\mathbf{1}) - S_\mu(x+\mathbf{1})$ is Schur positive, if and only if $P_\lambda(x+\mathbf{1}) - P_\mu(x+\mathbf{1})$ is Jack positive.

Theorem (Cuttler–Greene–Skandera '11, Sra '16)

Let $|\lambda| = |\mu|$. Then λ majorizes (=dominates) μ if and only if

$$S_\lambda(x) - S_\mu(x) \geqslant 0, \quad \forall x \in [0,\infty)^n.$$

Theorem (Khare–Tao '18)

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