Math 403:01 # Homework assignment #5 **March 24, 2010**

Due at the beginning of class, Wednesday, March 31, 2010

Please read section 2.4 of the textbook and begin section 2.5.

Solve these textbook problems (4 points each):

2.4 (page 133): 10, 13, 20, 21.

The Rewards of Theory

It may be difficult to understand the significance of the varied characterizations of analytic functions that we've discussed. These are powerful tools for working with functions and give amazing results. The two problems below are *easy* if you use the correct tools.

A. (10 points) Suppose f is a complex-valued function defined on all of $\mathbb{C}$, and that f has the following strange property:

> If z_0 is any complex number, then there is a sequence of complex numbers $\{a_n\}_{n\geq 0}$ so that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is valid for $|z-z_0| < 12$: that is, the infinite series converges, and its sum is $f(z)$ for z's whose distance to z_0 is less than 12. Note that the sequence of coefficients can *depend on* z_0: they may vary from one value of z_0 to another.

Prove that the radius of convergence of each power series is actually ∞ and that the sum of the series at each z is $f(z)$.

Remark I don't know how to prove this without the big results of complex analysis (and no one I've asked does either). The problem is almost simple using the big results with *no computation needed!*

B. (10 points) Define a function $F(z)$ by $F(z) = \int_0^1 \frac{e^{zt}}{1+t^2}\,dt$. Verify, by separating the integrand into its real and imaginary parts if necessary, that the integral is the integral of a continuous function over the interval $[0,1]$ for any complex number z and therefore $F(z)$ is well-defined.* Prove that $F(z)$ is analytic.

Hint Notice that, as a function of z with t fixed in $[0,1]$, $\frac{e^{zt}}{1+t^2}$ is certainly analytic. Use Morera's Theorem, and interchange the two integrals. Functions like this arise in the study of Fourier and Laplace transforms.

* **Warning** The only value I can compute explicitly of this function is $F(0) = \frac{\pi}{4}$. The other values are quite inaccessible to me. `Maple` again declares the integral is a complicated sum of various special functions that I never learned about. So don't compute anything!