Math 403:01 # Homework assignment #4 **February 10, 2010**

Due at the beginning of class, Monday, March 22, 2010

Please read sections 2.2, 2.3, and 2.4 of the textbook. This is the core of the course, and knowing this material is the gateway to success in the majority of what follows.

Solve these textbook problems (4 points each):

2.2 (page 103): 11, 18

2.3 (page 116): 4, 6, 13, 14

A. (10 points) For what simple closed curves γ is $\displaystyle\int_\gamma \frac{dz}{z^2+z+1} = 0$?

Hint Partial fractions, linearity, thought.

From *Basic Complex Analysis* by Jerrold Marsden

B. (10 points) In this problem, the curve C is the boundary of the circle of radius 1 centered at 0 oriented counterclockwise.

a) Evaluate $\displaystyle\int_C z^n\,dz$ (here n is an integer: positive, negative, or 0).

Remark Many, many of these are 0!

b) Evaluate $\displaystyle\int_C \left(z+\frac{1}{z}\right)^n \frac{dz}{z}$ (here n is a positive integer).

Hint Expand, compute.

c) Show that $\displaystyle\int_0^{2\pi} \sin^{2n}\theta\,d\theta = \int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$.

Remark You only need to check one of these, since the shapes are the same. Just "encode" an integral in a complex fashion. The "encode" is a slight joke since the integrals are attributed to John Wallis (1616–1703), a mathematician who was also a cryptographer for various English government organizations. The values of the integrals can be used to prove the Wallis product expansion for π (*not* part of the homework assignment!):

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots$$

From *Classical Complex Analysis* by Liang-shin Hahn and Bernard Epstein

The other side has an interesting quote from one of the "founders".

What's below is copied from Reinhold Remmert's wonderful book, *Theory of Complex Functions*. The book discusses both the history and the methods of complex analysis. It notes the errors committed and the variant approaches which were tried. Remmert writes that, in spite of this letter, the first publication of what is now recognized as Cauchy's Theorem and related results was done by Cauchy (who was trying to understand how to describe incompressible fluids mathematically!). This was around the same time as Green's important results. See how much you can recognize in this translation of Gauss's remarks.

> GAUSS wrote to BESSEL on December 18, 1811: "What should we make of $\int \varphi x \cdot dx$ for $x = a + bi$? Obviously, if we're to proceed from clear concepts, we have to assume that x passes, via infinitely small increments (each of the form $\alpha + i\beta$), from that value at which the integral is supposed to be 0, to $x = a + bi$ and from that then all the $\varphi x \cdot dx$ are summed up. In this way the meaning is made precise. But the progression of the x values can take place in infinitely many ways: Just as we think of the realm of all real magnitudes as an infinite straight line, so we can envision the realm of all magnitudes, real and imaginary, as an infinite plane wherein every point which is determined by an abscissa a and an ordinate b represents as well the magnitude $a+bi$. The continuous passage from one value of x to another $a+bi$ accordingly occurs along a curve and is consequently possible in infinitely many ways. But I maintain that the integral $\int \varphi x \cdot dx$ computed via two such passages always gets the same value as long as $\varphi x = \infty$ never occurs in the region of the plane enclosed by the curves describing these two passages. This is a very beautiful theorem whose not-so-difficult proof I will give when an appropriate occasion comes up. It is closely related to other beautiful truths having to do with developing functions in series. The passage from point to point can always be carried out without ever touching one where $\varphi x = \infty$. However, I demand that these points be avoided lest the original basic conception of $\int \varphi x \cdot dx$ lose its clarity and lead to contradictions. Moreover, it is also clear from this how a function generated by $\int \varphi x \cdot dx$ could have several values for the same values of x, depending on whether a point where $\varphi x = \infty$ is gone around not at all, once, or several times. If, for example, we define $\log x$ via $\int \frac{1}{x} dx$ starting at $x = 1$, then arrive at $\log x$ having gone around the point $x = 0$ one or more times or not at all, every circuit adds the constant $+2\pi i$ or $-2\pi i$; thus the fact that every number has multiple logarithms becomes quite clear."