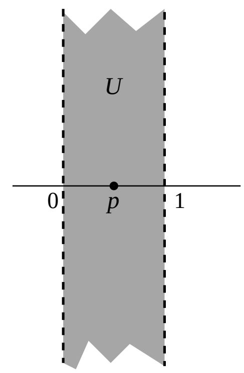
(20) 1. `Maple` reports that $\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+5)^2}\, dx = \frac{\pi}{16}$. Verify this statement using the Residue Theorem. Show clearly any contour of integration and any residue computation. Explain why the limiting value of certain integrals is 0.

(20) 2. a) For which complex numbers z is $z^2 = |z|^2$? For which complex numbers z is $z^2 = i|z|^2$?

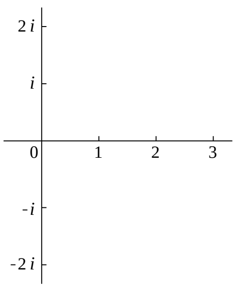
b) Find a specific complex number z so that $|\sin z| > 10$.

c) Find all values of $(1+i)^i$ in rectangular form ($x+iy$ where x and y are values of standard functions).

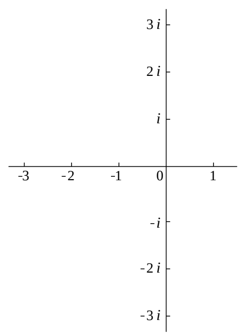
(20) 3. In this problem, U is the connected open set defined by $0 < \mathrm{Re}(z) < 1$ and $p = \frac{1}{2}$ as shown to the right.
In each part below, draw the image of U and p under the indicated mapping on the axes given. Label and describe the boundary curves as precisely as you can on each graph.



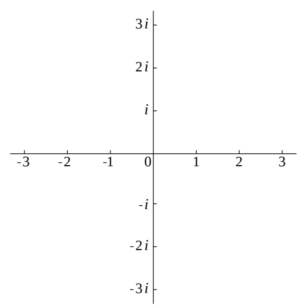
a) $f(z) = \frac{1}{z}$



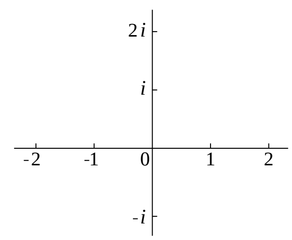
b) $g(z) = z^2$



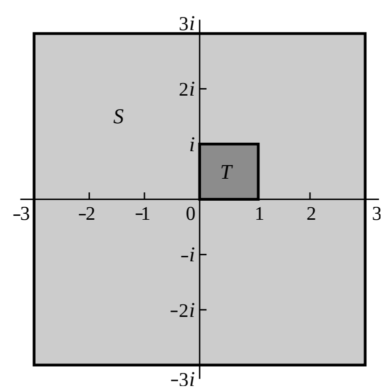
c) $h(z) = e^z$



d) $k(z) = iz$



(20) 4. Suppose T is the closed square with corners 0, 1, i, and $1+i$, and S is the closed square with corners $3+3i$, $-3+3i$, $-3-3i$, and $3-3i$. Suppose also that f is any entire function.
If M is the maximum of $|f''(z)|$ on T and N is the maximum of $|f(z)|$ on S, show that $M \leq \frac{1}{2}N$.
Hint Begin by writing a complex variables formula connecting f'' and f.



(20) 5. Suppose $f(z) = z^5 + 5z^2 + e^z$. How many zeros (counting multiplicity) does f have in the annular region $1 < |z| < 2$?

(20) 6. **True or false** If true, give a brief explanation. If false, correct what is written and give an example to show why the original statement is false. *Read the statements carefully!*

a) If $f(z)$ is entire, the mapping $w = f(z)$ preserves angles.

b) If $f(z)$ is analytic in a connected open set U and C is a simple closed curve in U, then $\int_C f(z)\,dz = 0$.

c) If $f(z)$ is analytic in a connected open set U and z_0 is in U, then the Taylor series for $f(z)$ centered at z_0 converges to $f(z)$ for all $z \in U$.

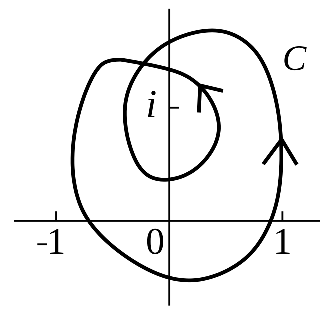
(20) 7. a) Suppose that $h(x,y)$ is harmonic, so $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$. Prove that $f = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}$ is analytic.

b) Verify that $k(x,y) = y + \cos x \cosh y$ is harmonic, and find one harmonic conjugate of $k(x,y)$. Write a function $f(z)$ of $z = x + iy$ alone which is analytic so that the real part of $f(z)$ is $k(x,y)$.

(20) 8. a) Suppose $f(z)$ is entire and these values of f and its derivatives are known:

$$f(0) = 2; \quad f'(0) = -3; \quad f''(0) = 2i;$$
$$f(i) = 7; \quad f'(i) = 4i; \quad f''(i) = 5.$$

Compute $\int_C \frac{f(z)}{z}\,dz$ and $\int_C \frac{f(z)}{(z-i)^2}\,dz$ if C is the curve shown to the right.



b) Evaluate $\int_Q \frac{dz}{e^z - 1}$ where Q is the circle of radius 9 and center 0, oriented positively (counterclockwise).

(20) 9. Suppose $f(z)$ is an entire function and there are positive real numbers A and B so that $|f(z)| \le A + B\sqrt{|z|}$. Prove that $f(z)$ is constant.

(20) 10. a) Find the first 2 non-zero terms of the Taylor series for $\sec(z)$ centered at $z = 0$. For which z's is the whole Taylor series guaranteed to converge to $\sec z$?

b) Find the first 2 non-zero terms of the Laurent series for $\sec(z)$ centered at $z = \frac{\pi}{2}$. For which z's is the whole Laurent series guaranteed to converge to $\sec z$?

Final Exam for Math 403, section 1

May 7, 2010

NAME ____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes, texts, or calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total Points Earned:		